8. [6 points] Suppose that $\int_{-3}^{8} f(x)dx = 5$. Use this information to determine the values for the constants $a, b,$ and k that you are certain will satisfy the definite integral $\int_a^b kf(2x)dx = 5$. Write your answers on the spaces provided. You do not need to show your work for this problem.

$a =$ **-1.5** $b =$ **4** $k =$ **2**

9. [6 points] Suppose $f(x) = f'(x) + 3$. Determine the EXACT value of $\int_0^1 e^x f'(x)dx$ given that $f(0) = 1$ and $f(1) = 4$. Be sure to show enough work to support your answer.

Solution: We use integration by parts, letting $u = e^x$ and $dv = f'(x)dx$ so that $du = e^x dx$ and $v = f(x)$. Then we have

$$\begin{aligned}
\int_0^1 e^x f'(x)dx &= e^x f(x)|_0^1 - \int_0^1 e^x f(x)dx \\
&= ef(1) - f(0) - \int_0^1 e^x (f'(x) + 3)dx \\
&= 4e - 1 - \int_0^1 e^x f'(x)dx - 3\int_0^1 e^x dx \\
2\int_0^1 e^x f'(x) &= 4e - 1 - 3e^x|_0^1 \\
\int_0^1 e^x f'(x) &= \frac{1}{2}((4e-1) - (3e-3)) = \frac{e+2}{2}
\end{aligned}$$

8. [8 points] Let f be a differentiable function with derivative f'. A table of values for f and f' is given below.

t	0	3	6	9
$f(t)$	1	2	7	5
$f'(t)$	1	4	-1	-2

Find the exact value of the following integrals.

a. [3 points] $\int_0^1 f'(3t)dt$.

Solution: Substituting $w = 3t, dw = 3dt$, we obtain

$$\int_{w(0)}^{w(1)} f'(w) \cdot \frac{1}{3} dw = \frac{1}{3} \int_0^3 f'(w) dw = \frac{1}{3} (f(3) - f(0)) = \frac{1}{3} (2 - 1) = \frac{1}{3}.$$

b. [5 points] $\int_3^9 t f''(t)dt$.

Solution: Using integration by parts with $u = t, dv = f''(t)dt$ (so that $du = dt, v = f'(t)$), we obtain

$$\int_3^9 t f''(t) dt = t \cdot f'(t)\Big|_3^9 - \int_3^9 f'(t) dt = \big(9f'(9) - 3f'(3)\big) - (f(9) - f(3))$$
$$= (9 \cdot (-2) - 3 \cdot 4) - (5 - 2) = -33.$$

3. [13 points] Use the table and the fact that

$$\int_0^{10} f(t)dt = 350$$

to evaluate the definite integrals below exactly (i.e., no decimal approximations). Assume $f'(t)$ is continuous and does not change sign between any consecutive t-values in the table.

t	0	10	20	30	40	50	60
$f(t)$	0	70	e^5	e^3	0	$\pi/2$	π

a. [4 points] $\int_0^{10} tf'(t)dt$

Solution:

$$\begin{aligned}\int_0^{10} tf'(t)dt &= tf(t)|_0^{10} - \int_0^{10} f(t)dt \\ &= 10f(10) - \int_0^{10} f(t)dt \\ &= 700 - 350 \\ &= 350.\end{aligned}$$

b. [4 points] $\int_{20}^{30} \frac{f'(t)}{f(t)}dt$

Solution:

$$\begin{aligned}\int_{20}^{30} \frac{f'(t)}{f(t)}dt &= \int_{f(20)}^{f(30)} \frac{1}{u}du \\ &= \ln|u||_{f(20)}^{f(30)} \\ &= \ln|f(30)| - \ln|f(20)| \\ &= 3 - 5 \\ &= -2.\end{aligned}$$

c. [5 points] $\int_{50}^{60} f(t)f'(t)\sin(f(t))dt$

Solution:

$$\begin{aligned}\int_{50}^{60} f(t)f'(t)\sin(f(t))dt &= \int_{f(50)}^{f(60)} w\sin(w)dw \\ &= -w\cos(w)|_{f(50)}^{f(60)} + \int_{f(50)}^{f(60)} \cos(w)dw \\ &= -\pi\cos(\pi) + \int_{\pi/2}^{\pi} \cos(w)dw \\ &= \pi - 1\end{aligned}$$

5. [16 points] Suppose that $f(x)$ is a function with the following properties:

- $\int_0^1 f(x)\,dx = -5$.
- $\int_0^3 f'(x)\,dx = 10$.
- The average value of $f(x)$ on $[1, 1.5]$ is -4.
- $\int_2^4 xf'(x)\,dx = 8$.

In addition, a table of values for $f(x)$ is given below.

x	0	1	2	3	4
$f(x)$	-7	-2	-2	m	0

Calculate (a)-(d) **exactly**. Show your work and do not write any decimal approximations.

a. [4 points] $m = 3$

Solution: Using the Fundamental Theorem in $\int_0^3 f'(x)\,dx = 10$ we get $f(3) - f(0) = 10$ which gives $m - (-7) = 10$ so $m = 3$.

b. [4 points] $\int_0^{1.5} f(x)\,dx = -7$

Solution:

$$\int_0^{1.5} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx = -5 + 0.5(-4) = -7$$

c. [4 points] $\int_2^4 f(x)\,dx = -4$

Solution: Using integration by parts in $\int_2^4 x f'(x)\,dx = 8$ we get $(4f(4) - 2f(2)) - \int_2^4 f(x)\,dx = 8$ which gives $\int_2^4 f(x)\,dx = 0 - 2(-2) - 8 = -4$.

d. [4 points] $\int_4^{16} f'(\sqrt{x})\,dx = 16$

Solution: Using the substitution $u = \sqrt{x}$ we get

$$\int_4^{16} f'(\sqrt{x})\,dx = \int_2^4 f'(u) \cdot 2u\,du = 2 \cdot 8 = 16$$

1. [13 points] Suppose that f is a twice-differentiable, function that satisfies

$$f(0) = 1 \qquad f(2) = 2 \qquad f(4) = 4 \qquad f'(2) = 3$$

$$\int_0^2 f(x)\,dx = 5 \qquad \int_2^4 f(x)\,dx = 7.$$

Evaluate the following integrals.

a. [4 points] $\int_0^2 x f'(x)\,dx$

Solution:

$$\int_0^2 x f'(x)\,dx = x f(x)\big|_0^2 - \int_0^2 f(x)\,dx = -1.$$

b. [4 points] $\int_{\sqrt{2}}^2 x f'(x^2)\,dx$

Solution:

$$\int_{\sqrt{2}}^2 x f'(x^2)\,dx = \frac{1}{2}\int_2^4 f'(u)\,du = 1.$$

c. [5 points] $\int_0^2 x^3 f'(x^2)\,dx$

Solution:

$$\int_0^2 x^3 f'(x^2)\,dx = \frac{1}{2}\int_0^4 u f'(u)\,du = \frac{1}{2}\left(u f(u)\big|_0^4 - \int_0^4 f(u)\,du \right) = 2.$$

1. [12 points] Suppose that f is a twice differentiable function with continuous second derivative. (That is, both f and f' are differentiable, and f'' is continuous.) The following table gives some values of f and f'.

x	0	1	2	3	4	5	6	e^3
$f(x)$	7	5	-1	0	11	-3	2	9
$f'(x)$	3	-4	-2	4	-5	0	-1	2

In parts (a) through (c) below, calculate the exact numerical value of the integral.
Write "NOT ENOUGH INFO" if there is not enough information to find the exact value.
Be sure to show your work clearly. No partial credit will be given for estimates.

a. [4 points] $\displaystyle\int_1^{e^3} \frac{f'(\ln x)}{x}\,dx$

Solution: The substitution $w = \ln(x)$ gives $dw = \frac{dx}{x}$ and

$$\int_1^{e^3} \frac{f'(\ln x)}{x}\,dx = \int_0^3 f'(w)\,dw = f(3) - f(0) = 0 - 7 = -7.$$

b. [4 points] $\displaystyle\int_0^4 x f''(x)\,dx$

Solution: Integration by parts with $u = x$ and $dv = f''(x)\,dx$ gives

$$\begin{aligned}\int_0^4 x f''(x)\,dx &= x f'(x)\Big|_0^4 - \int_0^4 f'(x)\,dx \\ &= \big(4 \cdot f'(4) - 0 \cdot f'(0)\big) - (f(4) - f(0)) \\ &= -20 - (11 - 7) = -24.\end{aligned}$$

c. [4 points] $\displaystyle\int_2^6 f'(x)\,[f(x)]^2\,dx$

Solution:
One Approach: substitution with $w = f(x)$ so $dw = f'(x)\,dx$

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = \int_{f(2)}^{f(6)} w^2\,dw = \frac{w^3}{3}\Big|_{-1}^{2} = \frac{8}{3} - \frac{-1}{3} = 3.$$

Another Approach: integration by parts with $u = f(x)^2$ and $dv = f'(x)\,dx$

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = [f(x)]^3\big|_2^6 - 2\int_2^6 f'(x)[f(x)]^2\,dx.$$

Moving the last term to the left hand side and dividing both sides of the resulting equation by 3 gives

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = [f(x)]^3\big|_2^6 = \frac{8 - (-1)}{3} = 3.$$