Parametrization and Polar Coordinates

1 Parametric Equations and Polar Coordinate

1.1 Parametric Equations

To represent the motion of a particle in the xy-plane we use two equations, $x = f(t)$ and $y = g(t)$, then at the time t the particle is at the location $(f(t), g(t)$. In this case, we call the equations for x and y the parametric equations, with parametrization t.
Remember that, in parametric equation, for the same line, the parametrization is not unique, and the different parametrization encodes two information:
1. Speed of the particle.
2. Direction of the motion.

1.1.1 Special Parametric Equations

- Parametric Equations for a Straight Line

  An object moving along a line through the point (x_0, y_0), with $dx/dt = a$ and $dy/dt = b$, has parametric equations $x = x_0 + at, y = y_0 + bt$. The slope of the line is $m = b/a$.

- Parametric Equations for a circle with radius k

  An object moving along a circle of radius k counterclockwise has parametric equations $x = k\cos(t), y = k\sin(t)$.

1.1.2 Slope and concavity of the curve

As we discussed in class, we can think of this as a result due to chain rule if we have that $y = F(x)$ as well.
But to summarize, we have the slope of the parametrized curve to be

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

and the concavity of the parametrized curve to be

$$\frac{d^2y}{dx^2} = \frac{(dy/dx)/dt}{dx/dt}$$

1.1.3 Speed and distance

The instantaneous speed of a moving object is defined to be

$$v = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(v_x)^2 + (v_y)^2}$$

. The quantity $v_x = dx/dt$ is the instantaneous velocity in the x-direction; $v_y = dy/dt$ is the instantaneous velocity in the y-direction. And we call that (v_x, v_y) to be the velocity vector.
Moreover, the distance traveled from time a to b is

$$\int_a^b v(t)dt = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$$

1.2 Polar Coordinate

Polar coordinates is the coordinates determined by specifying the distance of the point to origin and the angle measured counterclockwise from positive x-axis to the line joining the line connecting the point and the origin.

1.2.1 Relation between Cartesian and Polar

Cartesian to Polar:

$$(x, y) \rightarrow (r = \sqrt{x^2 + y^2}, \theta) \text{ (Here we have that } \tan\theta = \frac{y}{x})$$

Note that θ does not have to be $\arctan(\frac{y}{x})$!
Polar to Cartesian:

$$(r, \theta) \rightarrow (x = r\cos\theta, y = r\sin\theta)$$

1.2.2 Slope, Arc length and Area in Polar Coordinates

By the relation $x = r\cos\theta, y = r\sin\theta$, given a curve $r = f(\theta)$, we have that $x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$, and thus are parametrized equations of parameter θ. Therefore we have that the slope of to be

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

The arc length from angle a to b is

$$\int_b^a \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}d\theta$$

Moreover, due to the fact that the area of the sector is $1/2r^2\theta$, we have that for a curve $r = f(\theta)$, with $f(\theta) \geq 0$, the area of the region enclosed is

$$\frac{1}{2}\int_a^b f(\theta)^2 d\theta$$