MATH 116 — PRACTICE FOR EXAM 2

Generated October 5, 2017

NAME: SOLUTIONS

INSTRUCTOR: ______________________ SECTION NUMBER: __________

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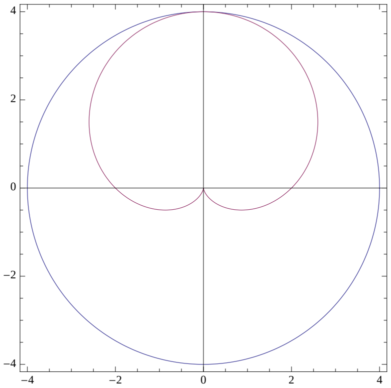
1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2010	2	2	cardioid	14	
Winter 2014	3	7	alpaca pool	8	
Fall 2014	2	8	crop circle	11	
Fall 2014	3	5	wild chickens	11	
Winter 2015	3	10	ladybugs2	12	
Fall 2015	2	4	moon	10	
Total				66	

Recommended time (based on points): 69 minutes

2. [14 points] The graph of the circle $r = 4$ and and the cardioid $r = 2\sin\theta - 2$ are shown below.



a. [3 points] Write a formula for the area inside the circle and outside the cardioid in the first quadrant.

Solution: Area of the quarter of a circle$= 4\pi$

Area of cardioid$=\int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$

Area$=4\pi - \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2\sin\theta - 2)^2 d\theta$

b. [7 points] At what angles $0 \leq \theta < 2\pi$ is the minimum value of the y coordinate on the cardioid attained? No credit will be given for answers without proper mathematical justification.

Solution:

$y(\theta) = (2\sin\theta - 2)\sin\theta$

$y'(\theta) = 2\cos\theta\sin\theta + (2\sin\theta - 2)\cos\theta = 4\cos\theta\sin\theta - 2\cos\theta$

Critical points $\quad (4\sin\theta - 2)\cos\theta = 0$

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$ then $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

Minimum y coordinate at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

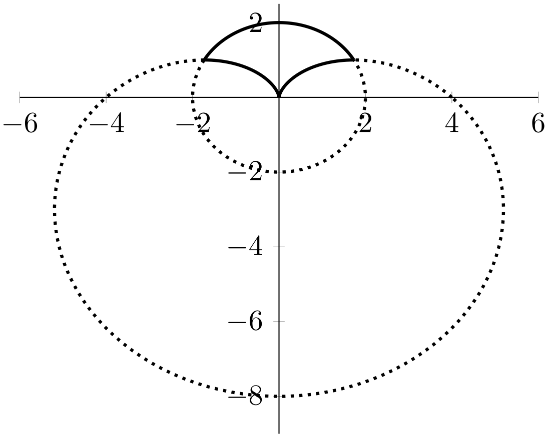
c. [4 points] Write an integral that computes the value of the length of the piece of the cardioid lying below the x-axis.

Solution:

$x(\theta) = (2\sin\theta - 2)\cos\theta \qquad x'(\theta) = 2\cos^2\theta - (2\sin\theta - 2)\sin\theta$

$L = \int_0^{\pi} \sqrt{(2\cos^2\theta - (2\sin\theta - 2)\sin\theta)^2 + (4\cos\theta\sin\theta - 2\cos\theta)^2} d\theta$

7. [8 points] Roy the alpaca is designing a pool and a deck for his family. The pool has the shape of a cardioid whose equation is given by $r = 4 - 4\sin(\theta)$ where r is in meters and θ is a number between 0 and 2π. The deck will be built in the region that lies inside the circle $x^2 + y^2 = 4$ and outside the cardioid. The deck is depicted in the figure as the region enclosed by the solid lines



a. [1 point] Write the equation for the circle $x^2 + y^2 = 4$ in polar coordinates.

Solution: $r^2 = 4$ so $r = 2$

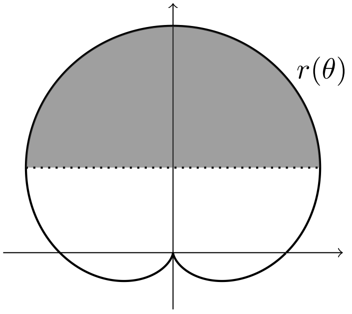
b. [2 points] Find the values of θ between 0 and 2π where the cardioid and the circle intersect.

Solution: Setting the two equations equal to each other we have $2 = 4 - 4\sin(\theta)$ thus $\sin(\theta) = \frac{1}{2}$. Therefore $\theta = \pi/6, 5\pi/6$.

c. [5 points] Write an expression involving integrals that gives the area of the region where the deck will be built. Do not evaluate your expression.

Solution: $\int_{\pi/6}^{5\pi/6} 2\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4 - 4\sin(\theta)^2\, d\theta = 4\pi/3 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4 - 4\sin(\theta)^2\, d\theta.$

8. [11 points] Franklin, your robot, uses the lasers on his satellites to burn strange shapes in local corn fields. One of these strange shapes is given by the polar equation $r(\theta) = 2 + 2\sin(\theta)$ where $r(\theta)$ is measured in kilometers. All of the corn **above** the line $y = \frac{3}{2}$ has been pecked away by a flock of wild chickens. The polar curve $r(\theta)$ (solid) and the line $y = \frac{3}{2}$ (dotted) are shown below. The portion of the corn field that has been pecked away is shaded below.



a. [6 points] Write an expression involving one or more integrals which gives the area of the shaded region. Do not evaluate any integrals. **Include units.**

Solution: First, we need to find the values of θ where $r(\theta)$ intersects the line $y = 3/2$. Since $y = r\sin(\theta)$, we have that

$$3/2 = [2 + 2\sin(\theta)]\sin(\theta)$$

This is a quadratic equation in $\sin(\theta)$, and the quadratic formula gives $\sin(\theta) = 1/2$ or $-3/2$. Since $\sin(\theta)$ is bounded between -1 and 1, $\sin(\theta) = 1/2$. This means that:

$$\theta = \pi/6 \qquad \text{or} \qquad 5\pi/6$$

The line $y = 3/2$ has polar equation $r_{\text{line}}(\theta) = \frac{3}{2\sin(\theta)}$. Therefore the area bounded between $r(\theta)$ and $r_{\text{line}}(\theta)$ is given by the integral:

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} r(\theta)^2 - r_{\text{line}}(\theta)^2 d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2 + 2\sin(\theta))^2 - \left(\frac{3}{2\sin(\theta)}\right)^2 d\theta \qquad \text{km}^2$$

b. [5 points] Write an expression involving one or more integrals which gives the perimeter of the shaded region. Do not evaluate any integrals. **Include units.**

Solution: To find the length of the portion of the line $y = 3/2$ passing through the shaded region, we can take the difference of the x-coordinates of the points where $r(\theta)$ intersects the line. Since $r_{\text{line}}(\theta) = \frac{3}{2\sin(\theta)}$, the x-coordinates are given by $x = r_{\text{line}}(\theta)\cos(\theta) = \frac{3\cos(\theta)}{2\sin(\theta)}$

$$\text{Length of segment} = \frac{3\cos(\pi/6)}{2\sin(\pi/6)} - \frac{3\cos(5\pi/6)}{2\sin(5\pi/6)} \approx 5.196 \qquad \text{km}$$

Now we need to calculate the portion of the perimeter lying on $r(\theta)$. For this we can use the polar perimeter formula

$$\text{Length over } r(\theta) = \int_{\pi/6}^{5\pi/6} \sqrt{(2+2sin(\theta))^2 + (2\cos(\theta))^2}d\theta \quad \text{km}$$

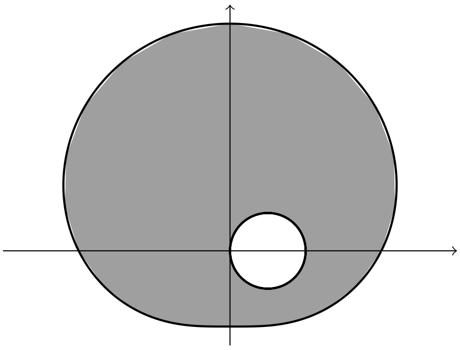
$$\text{Perimeter} = \frac{3\cos(\pi/6)}{2\sin(\pi/6)} - \frac{3\cos(5\pi/6)}{2\sin(5\pi/6)} + \int_{\pi/6}^{5\pi/6} \sqrt{(2+2sin(\theta))^2 + (2\cos(\theta))^2}d\theta \quad \text{km}$$

5. [11 points] Franklin's robot army is surrounding you!

a. [6 points] Consider the polar curves

$$r = \cos(\theta) \qquad\qquad r = \sin(\theta) + 2$$

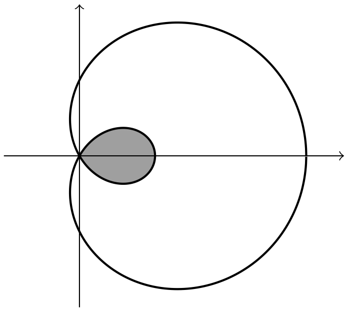
Franklin's robot army occupies the shaded region between these two curves. Write an expression involving integrals that gives the **area** occupied by Franklin's robot army. Do not evaluate any integrals.



Solution:

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\sin(\theta)+2)^2 d\theta - \frac{1}{2}\int_0^{\pi} (\cos(\theta))^2 d\theta$$

b. [5 points] Your friend, Kazilla, pours her magic potion on the ground. Suddenly, a flock of wild chickens surrounds you. The chickens occupy the shaded region enclosed within the polar curve $r = 1 + 2\cos(\theta)$ as shown below. Write an expression involving integrals that gives the **perimeter** of the region occupied by the flock of wild chickens. Do not evaluate any integrals.



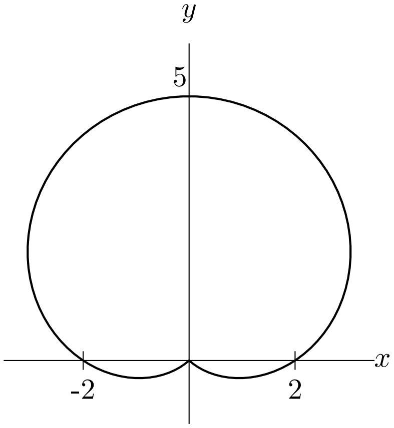
Solution: We use the arc length formula:

$$\text{Arc Length} = \int_a^b \sqrt{(r(\theta))^2 + (r'(\theta))^2} d\theta$$

Note that $r'(\theta) = -2\sin(\theta)$. Also, the shaded region of lies between $\theta = 2\pi/3$ and $\theta = 4\pi/3$ (you can see this by setting $r(\theta) = 0$, and testing that $r(\pi) = -1$, so it lies on the boundary of the shaded region.)

$$\text{Arc Length} = \int_{2\pi/3}^{4\pi/3} \sqrt{(1+2\cos(\theta))^2 + (-2\sin(\theta))^2} d\theta$$

10. [12 points] Vic is watching the ladybugs run around in his garden. His garden is in the shape of the outer loop of a cardioid with polar equation $r = 2 + 3\sin\theta$ where r is measured in meters and θ is measured in radians. The outline of the garden is pictured below for your reference. At a time t minutes after he begins watching, Apple, his favorite red ladybug, is at the xy-coordinate $(\sin^2 t, \cos^2 t)$, and Emerald, his prized green ladybug, is at the xy-coordinate $(-\cos(2t), \sin(2t) + 1.5)$. Vic watches the ladybugs for 2π minutes.



Using the information above, circle the correct answer for each part below. There is only one correct answer for each part. You do not need to show your work.

a. [3 points] Which of the following integrals gives the area of the garden?

Solution:

A) $\displaystyle\frac{1}{2}\int_{0}^{\pi} (2 + 3\sin\theta)^2 d\theta$

B) $\displaystyle\frac{1}{2}\int_{\arcsin(\frac{2}{3})}^{\pi+\arcsin(\frac{2}{3})} (2 + 3\sin\theta)^2 d\theta$

$$\boxed{\textbf{C)}\ \frac{1}{2}\int_{-\arcsin(\frac{2}{3})}^{\pi+\arcsin(\frac{2}{3})} (2 + 3\sin\theta)^2 d\theta}$$

D) $\displaystyle\frac{1}{2}\int_{0}^{2\pi} (2 + 3\sin\theta)^2 d\theta$

E) $\displaystyle\frac{1}{2}\int_{2\pi-\arcsin(\frac{2}{3})}^{4\pi-\arcsin(\frac{2}{3})} (2 + 3\sin\theta)^2 d\theta$

b. [3 points] Which of the following is **not** true about Apple while Vic is watching?

Solution:

A) Apple runs through the point $(\frac{1}{2}, \frac{1}{2})$ more than once.

B) Apple crosses the path made by Emerald exactly 4 times.

C) Apple's speed is zero at least once.

D) Apple does not leave the garden.

$\boxed{\textbf{E)}\ \text{Apple is moving faster than Emerald for some of the time.}}$

c. [3 points] How far does Emerald run while Vic is watching?

Solution:

A) π meters

B) 2π meters

[✓ **C)** 4π meters]

D) $\sqrt{8}\pi$ meters

E) 8π meters

d. [3 points] After the 2π minutes, Vic stops watching. Apple runs from the point $(x, y) = (0, 1)$ in the positive y-direction with speed of $g(T) = 5Te^{-T}$, T seconds after Vic stops watching. Which of the following is true?

Solution:

[✓ **A)** Apple leaves the garden eventually, but never runs further than 5 meters total.]

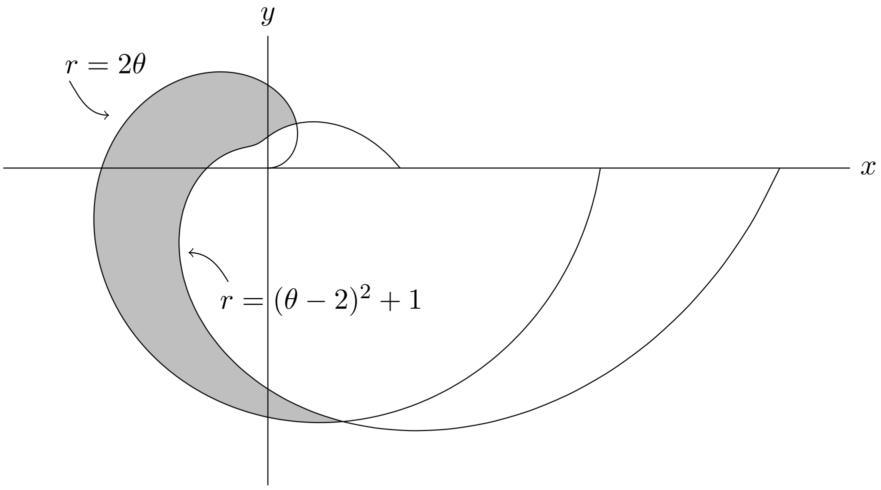
B) Apple's speed is always increasing after Vic stops watching.

C) If given enough time, Apple would eventually be more than 1000 meters from the garden.

D) Apple is still in the garden 5 minutes after Vic stops watching.

E) Apple changes direction, eventually.

4. [10 points] The visible portion of the strangely-shaped moon of the planet Thethis during its waxing crescent phase is in the shape of the region bounded between the polar curves $r = 2\theta$ and $r = (\theta - 2)^2 + 1$. The region is pictured below. Assume x and y are measured in thousands of miles.



a. [6 points] Write an expression involving integrals which gives the area of the visible portion of this moon. Include the units of the integral in your answer. Do not evaluate any integrals.

Solution: The two curves intersect when $\theta = 1, 5$. Therefore the area of the moon is

$$\left(\int_1^5 \frac{1}{2}(2\theta)^2 d\theta - \int_1^5 \frac{1}{2}((\theta-2)^2+1)^2 d\theta\right) \text{ (thousand miles)}^2$$
$$= \left(\int_1^5 \frac{1}{2}(2\theta)^2 d\theta - \int_1^5 \frac{1}{2}((\theta-2)^2+1)^2 d\theta\right) \text{ million miles}^2.$$

b. [4 points] Find the slope of the tangent line to the polar curve $r = (\theta - 2)^2 + 1$ at $\theta = \pi$.

Solution: Converting to parametric equations, we have

$$x = r\cos\theta = ((\theta-2)^2+1)\cos\theta$$
$$y = r\sin\theta = ((\theta-2)^2+1)\sin\theta$$

Thus

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{\left.\frac{dy}{d\theta}\right|_{\theta=\pi}}{\left.\frac{dx}{d\theta}\right|_{\theta=\pi}} = \frac{2(\pi-2)\sin\pi + ((\pi-2)^2+1)\cos\pi}{2(\pi-2)\cos\pi - ((\pi-2)^2+1)\sin\pi} = \frac{(\pi-2)^2+1}{2(\pi-2)}.$$