Math 116 — Practice for Exam 3

Generated November 18, 2020

Name: SOLUTIONS

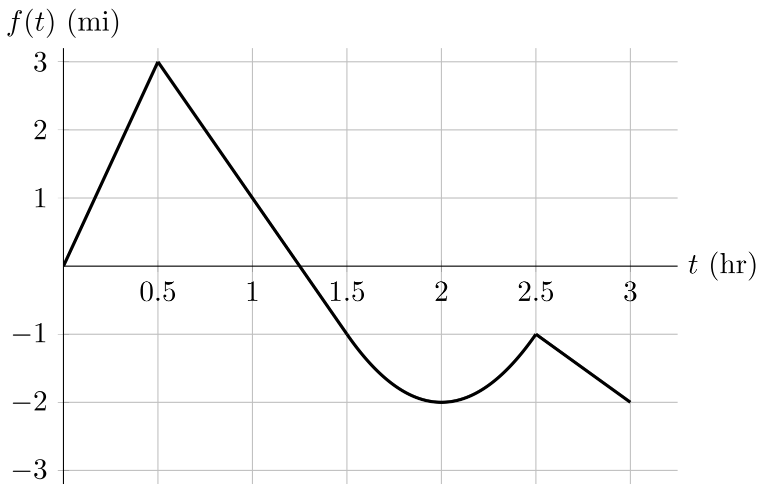
Instructor: ______________________ Section Number: __________

1. This exam has 3 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2017	2	6	Venice Beach	13	
Fall 2011	2	2		12	
Winter 2011	2	5	ant	10	
Total				35	

Recommended time (based on points): 32 minutes

6. [13 points] Anderson and Glen decide to take a road trip starting from Venice Beach. They have no worries about getting anywhere quickly, as they enjoy each other's company, so they take a very inefficient route. Suppose that Venice Beach is located at $(0,0)$ and that Anderson and Glen's position (x,y) (measured in miles) t hours after leaving Venice Beach is given by a pair of parametric equations $x=f(t)$, $y=g(t)$. A graph of $f(t)$ and a formula for $g(t)$ are given below. Note that $f(t)$ is linear on the intervals $[0,0.5]$, $[0.5,1.5]$, and $[2.5,3]$.



$$g(t)=-t^3+5t^2-3t$$

Note: For each of the following, your final answer should **not** involve the letters f and g.

a. [2 points] If their roadtrip last 3 hours, what are the $x-$ and $y-$ coordinates of their final destination?

Solution: Note that at time $t=3$, we have $x=f(3)=-2$ and $y=g(3)=9$. So the coordinates of their final destination are $(-2,9)$.

b. [3 points] At what speed are they traveling 2 hours into their trip?

Solution: We have $\left.\frac{dx}{dt}\right|_{t=2}=f'(2)=0$ and $\left.\frac{dy}{dt}\right|_{t=2}=g'(2)=5.$
So their speed at time $t=2$ is $\sqrt{0^2+5^2}=5$ miles per hour.

c. [4 points] Write, but do not compute, an expression involving one or more integrals that gives the distance they traveled, in miles, in the first **half** hour of their trip.

Solution: On the interval $(0,0.5)$, we see that $f(t)=6t$, so on this interval, we have

$$f'(t)=6 \quad \text{and} \quad g'(t)=-3t^2+10t-3.$$

The parametric arc length formula then implies that the distance they travelled from $t=0$ to $t=0.5$ is $\int_0^{0.5}\sqrt{(6)^2+(-3t^2+10t-3)^2}\,dt$ miles.

d. [4 points] Write down a pair of parametric equations using the parameter s for the line tangent to their path at $t=2.75$ hours.

Solution: Note that

$$f(2.75)=-1.5,\quad \left.\frac{df}{dt}\right|_{t=2.75}=-2,\quad g(2.75)=8.765625,\quad \text{and} \quad \left.\frac{dg}{dt}\right|_{t=2.75}=1.8125$$

There are many possible parametrizations. There is no need to have this match with the parameter t from earlier, so the answer below has the line passing through $(-1.5,8.765625)$ at $s=0$.

Answer: $x(s)=$ $-2s-1.5$ and $y(s)=$ $1.8125s+8.765625$

2. [12 points] Consider a particle whose trajectory in the xy-plane is given by the parametric curve defined by the equations

$$x(t) = t^4 - 4t^2, \qquad y(t) = t^2 - 2t,$$

for $-3 \le t \le 3$. Show all your work to receive full credit.

a. [3 points] Is there any value of t at which the particle ever comes to a stop? Justify.

Solution: No. For the particle to come a stop, its velocity in both the x- and y-direction must be zero. We have that

$$\frac{dx}{dt} = 4t^3 - 8t = 4t(t^2 - 2) = 0$$

at $t = 0, \pm\sqrt{2}$ and

$$\frac{dy}{dt} = 2t - 2 = 0$$

at $t = 1$. Since there are no times at which $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are simultaneously zero, the particle never comes to a stop.

b. [2 points] For what values of t does the path of the particle have a vertical tangent line?

Solution: Vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. From the above calculation, this is true at $t = 0, \pm\sqrt{2}$.

c. [3 points] What is the lowest point (x, y) on the curve?

Solution: We want to minimize the value of the y-coordinate over $-3 \le t \le 3$. The only critical point for $y(t)$ was found above at $t = 1$. Since $\left.\frac{dy}{dt}\right|_{t=0} = -2 < 0$ and $\left.\frac{dy}{dt}\right|_{t=2} = 2 > 0$, the First Derivative Test tells us that $t = 1$ is a local minimum, and thus a global minimum since it is the only critical point on the given interval. The lowest point on the curve is thus $(x(1), y(1)) = (-3, -1)$.

d. [2 points] At what values of t does the particle pass through the origin?

Solution: We set $x(t) = 0$ and $y(t) = 0$ and solve for t.

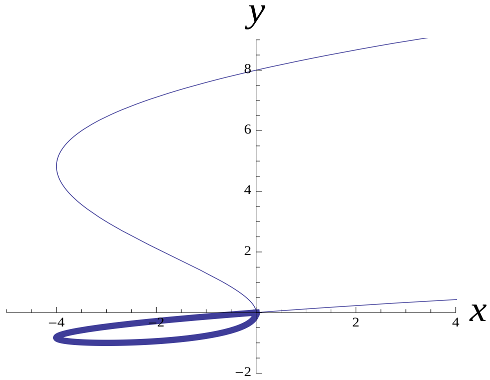
$$x(t) = t^4 - 4t^2 = t^2(t^2 - 4) = t^2(t-2)(t+2) = 0$$

gives that $t = -2, 0, 2$, while

$$y(t) = t^2 - 2t = t(t-2) = 0$$

gives $t = 0, 2$.
Thus, the particle passes through the origin at times $t = 0$ and $t = 2$.

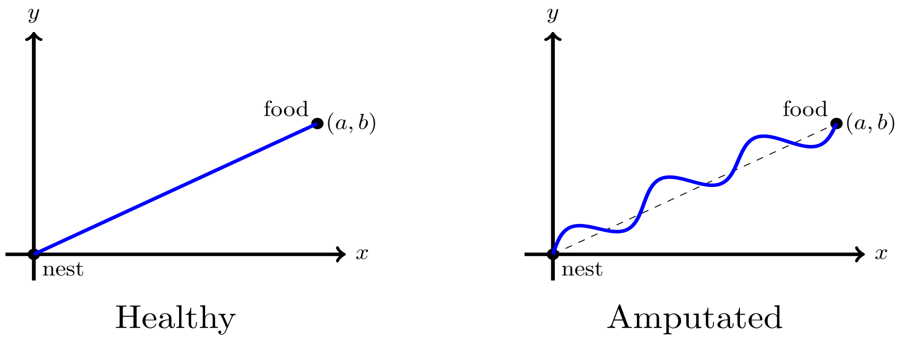
e. [2 points] The graph of the curve traced by these parametric equations is shown below. Find an expression for the length of the closed loop marked in the graph.



Solution: From the given graph and above calculation, we know that the loop is traced out over the time interval $0 \le t \le 2$. The arclength of the loop is given by

$$\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(4t^3 - 8t)^2 + (2t-2)^2}\, dt.$$

5. [10 points]
When an ant finds food, it leaves a trail of chemicals from its nest to the food source. Other ants follow this chemical trail using their two antennae. When an ant possesses both antennae, it will walk in a straight line to the food. If you remove (amputate) the left antenna of an ant, it will walk in a pattern like the one shown in the second figure.



a. [4 points] Write a parametric equation for the path of a healthy ant that starts at its nest at (0, 0) when $t = 0$ and arrives at the food at (a, b) when $t = 1$.

Solution: $x(t) = at$ and $y(t) = bt$.

b. [6 points] Suppose the parametric equation for the amputated ant is given by

$$x = x(t) \qquad y = y(t).$$

Assume the ant starts walking at $t = 0$, arrives at the food at $t = 1$, and never pauses or backtracks. For each blank below, determine whether the number on the left is greater than, less than, or equal to the number on the right, and fill the blank with the symbol $>$, $<$, or $=$ respectively. **Justify your answers**.

Solution:

$$\frac{y'(1)}{x'(1)} \qquad > \qquad 0$$

Answer 1: The slope of the tangent line to the curve at (a, b) is positive

Answer 2: The quotient is undefined since the ant stopped.

$$x'(c) \quad (\text{for any } 0 < c < 1) \qquad > \qquad 0$$

The ant is always moving to the right

$$\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}dt \qquad > \qquad \sqrt{a^2 + b^2}$$

The length of the line is shorter than the length of the curve