Final Exam Summary (Everything, formula)

1 Integral

2 Riemann Sums

1. $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$**(Limit of Right-hand sum RIGHT(n))**

2. $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$**(Limit of Left-hand sum LEFT(n))**

3. $\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} f(\frac{x_i+x_{i+1}}{2})\Delta x$**(Limit of Mid sum MID(n))**

4. $\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{f(x_i)+f(x_{i+1})}{2}\Delta x$**(Limit of Trapezoid sum TRAP(n))**

5. $\Delta(x) = \dfrac{b-a}{n}$

6. $\frac{LEFT(n)+RIGHT(n)}{2} = TRAP(n)$

7. $MID(n) \neq TRAP(n)$

8. Error estimation: $|LEFT(n) - f(x)| < |LEFT(n) - RIGHT(n)| = (f(b) - f(a))\Delta x$. This usually gives a bound for n.

2.1 Properties of Riemann sums:

1. If the graph of f is increasing on $[a, b]$, then $LEFT(n) \leq \int_a^b f(x)dx \leq RIGHT(n)$

2. If the graph of f is decreasing on $[a, b]$, then $RIGHT(n) \leq \int_a^b f(x)dx \leq LEFT(n)$

3. If the graph of f is concave up on $[a, b]$, then $MID(n) \leq \int_a^b f(x)dx \leq TRAP(n)$

4. If the graph of f is concave down on $[a, b]$, then $TRAP(n) \leq \int_a^b f(x)dx \leq MID(n)$

2.2 Properties of Definite Integrals

1. $int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\int_b^a f(x)dx + \int_c^b f(x)dx = \int_c^a f(x)dx$
3. $\int_b^a (f(x) \pm g(x))dx = \int_b^a f(x)dx \pm \int_b^a g(x)dx$
4. $\int_b^a cf(x)dx = c\int_b^a f(x)dx$
5. Symmetry due to the oddity of the function.
6. Average value of function $f(x)$ in $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

Theorem 2.1. The Fundamental Theorem of Calculus:
If f is continuous on interval $[a, b]$ and $f(t) = F'(t)$, then $\int_a^b f(t)dt = F(b) - F(a)$. **Second FTC (Construction theorem for Antiderivatives)** If f is a continuous function on an interval, and if a is any number in that interval then the function F defined on the interval as follows is an antiderivative of f:

$$F(x) = \int_a^x f(t)dt$$

1. $\int Cdx = 0$
2. $\int kdx = kx + C$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, (n \neq -1)$
4. $\int \frac{1}{x}dx = \ln|x| + C$
5. $\int e^x dx = e^x + C$
6. $\int \cos x dx = \sin x + C$
7. $\int \sin x dx = -\cos x + C$

Properties of antiderivatives:

1. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
2. $\int cf(x)dx = c\int f(x)dx$

2.3 Integration Techniques

1. Guess and Check
2. Substitution $du = f(x)'dx$ if $u = f(x)$
3. By parts $\int udv = uv - \int vdu$
4. Partial fractions $\frac{p(x)}{(x+c_1)^2(x+c_2)(x^2+c_3)} = \frac{A}{x+c_1} + \frac{B}{(x+c_1)^2} + \frac{C}{x+c_2} + \frac{Dx+E}{x^2+c_3}$

3 Find Area/Volumes by slicing

1. Compute the area: Think about slicing the area into parallel line segments.

2. Disk Method:
   Horizontal axis of revolution (x-axis): $V = \int_a^b \pi(f(x)^2 - g(x)^2)dx$
   Vertical axis of revolution (y-axis): $V = \int_a^b \pi(f(y)^2 - g(y)^2)dy$

3. Shell Method:
   Horizontal axis of revolution (x-axis): $V = \int_a^b 2\pi y(f(y) - g(y))dy$
   Vertical axis of revolution (y-axis): $V = \int_a^b 2\pi x(f(x) - g(x))dx$

3.1 Mass

The basic formula we are doing is:

1. One dimensional: $M = \delta l$ where M is the total mass, δ is the density, l is line.

2. Two dimensional: $M = \delta A$ where M is the total mass, δ is the density, A is Area.

3. Three dimensional (real world): $M = \delta V$ where M is the total mass, δ is the density, V is Volume.

3.2 Work

Key formula we are using:
Work done = Force $\cdot$ Distance or $W = F \cdot s$
Integration version: $W = \int_a^b F(x)dx$

3.3 L'Hopital's rule

LHopitals rule: If f and g are differentiable and (below a can be $\pm\infty$)
i)$f(a) = g(a) = 0$ for finite a,
Or ii)$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$,
Or iii)$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

3.4 Dominance

We say that g dominates f as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.

4 Improper integral

There are two types of improper integral.

- The first case is where we have the limit of the integration goes to infinity, i.e. $\lim_{b\to\infty}\int_a^b f(x)dx$.
- The integrand goes to infinity as $x \to a$.

4.1 Converges or diverges?

1. Check by definition, this means check the limit directly.
2. p-test.

	$p<1$	$p=1$	$p>1$
Type I : $\int_a^\infty \frac{1}{x^p}dx$	diverges	$=\ln x \big\|_a^\infty \Rightarrow$ diverges	converges
Type II : $\int_0^a \frac{1}{x^p}dx$	converges	$=\ln x \big\|_0^a \Rightarrow$ diverges	diverges

3. Exponential decay test.
$$\int_0^\infty e^{-ax}dx$$
converges for $a>0$.

4. Comparison test.
If $f(x) \geq g(x) \geq 0$ on the interval $[a,\infty]$ then,
   - If $\int_a^\infty f(x)dx$ converges then so does $\int_a^\infty g(x)dx$.
   - If $\int_a^\infty g(x)dx$ diverges then so does $\int_a^\infty f(x)dx$.

5. Limit Comparison theorem.
Limit Comparison Test. If $f(x)$ and $g(x)$ are both positive on the interval $[a,b)$ where b could be a real number or infinity. and
$$\lim_{x\to b}\frac{f(x)}{g(x)} = C$$
such that $0 < C < \infty$ then the improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are either both convergent or both divergent.

5 Probability

5.1 PDF and CDF

Definition 5.1. A function $p(x)$ is a **probability density function** or PDF if it satisfies the following conditions

- $p(x) \geq 0$ for all x.
- $\int_{-\infty}^{\infty} p(x) = 1$.

Definition 5.2. A function $P(t)$ is a **Cumulative Distribution Function** or cdf, of a density function $p(t)$, is defined by $P(t) = \int_{-\infty}^{t} p(x)dx$, which means that $P(t)$ is the antiderivative of $p(t)$ with the following properties:

- $P(t)$ is increasing and $0 \leq P(t) \leq 1$ for all t.
- $\lim_{t\to\infty} P(t) = 1$.
- $\lim_{t\to-\infty} P(t) = 0$.

Moreover, we have $\int_{a}^{b} p(x)dx = P(b) - P(a)$.

5.2 Probability, mean and median

Probability

Let us denote X to be the quantity of outcome that we care (X is in fact, called the random variable). $\mathbb{P}\{a \leq X \leq b\} = \int_{a}^{b} p(x)dx = P(b) - P(a)$
$\mathbb{P}\{X \leq t\} = \int_{-\infty}^{t} p(x)dx = P(t)$
$\mathbb{P}\{X \geq s\} = \int_{s}^{\infty} p(x)dx = 1 - P(s)$

The mean and median

Definition 5.3. A **median** of a quantity X is a value T such that the probability of $X \leq T$ is $1/2$. Thus we have T is defined such that $\int_{-\infty}^{T} p(x)dx = 1/2$ or $P(T) = 1/2$.

Definition 5.4. A **mean** of a quantity X is the value given by

$$Mean = \frac{\text{Probability of all possible quantity}}{\text{Total probability}} = \frac{\int_{-\infty}^{\infty} xp(x)dx}{\int_{-\infty}^{\infty} p(x)dx} = \frac{\int_{-\infty}^{\infty} xp(x)dx}{1} = \int_{-\infty}^{\infty} xp(x)dx.$$

Normal Distribution

Definition 5.5. A normal distribution has a density function of the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean of the distribution and σ is the standard deviation, with $\sigma > 0$. The case $\mu = 0$, $\sigma = 1$ is called the standard normal distribution.

6 Sequences and Series

6.1 Sequence

If a sequence s_n is bounded and monotone, it converges.

6.2 Series

Convergence Properties of Series:

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and if k is a constant, then
   $\sum_{n=1}^{\infty}(a_n + b_n)$ converges to$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
   $\sum_{n=1}^{\infty} ka_n$ converges to $k\sum_{n=1}^{\infty} a_n$

2. Changing a finite number of terms in a series does not change convergence,

3. If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges. (**!**)

4. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges if $k \neq 0$.

Moreover, there are several test to determine if a series is convergent.

1. **The Integral Test**
   Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.
   a. If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ an converges.
   b. If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ an diverges.

2. **p-test**
   The p-series $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.

3. **Comparison Test**
   Suppose $0 \leq a_n \leq b_n$ for all n beyond a certain value.
   a. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
   b. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

4. **Limit Comparison Test**
   Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} a_n/b_n = c$ where $c > 0$, then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

5. **Convergence of Absolute Values Implies Convergence**
   If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

6. **The Ratio Test** For a series $\sum_{n=1}^{\infty} a_n$, suppose the sequence of ratios $|a_{n+1}|/|a_n|$ has a limit: $\lim_{n\to\infty} |a_{n+1}|/|a_n| = L$, then
   - If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
   - If $L > 1$, or if L is infinite, then $\sum_{n=1}^{\infty} a_n$ diverges.

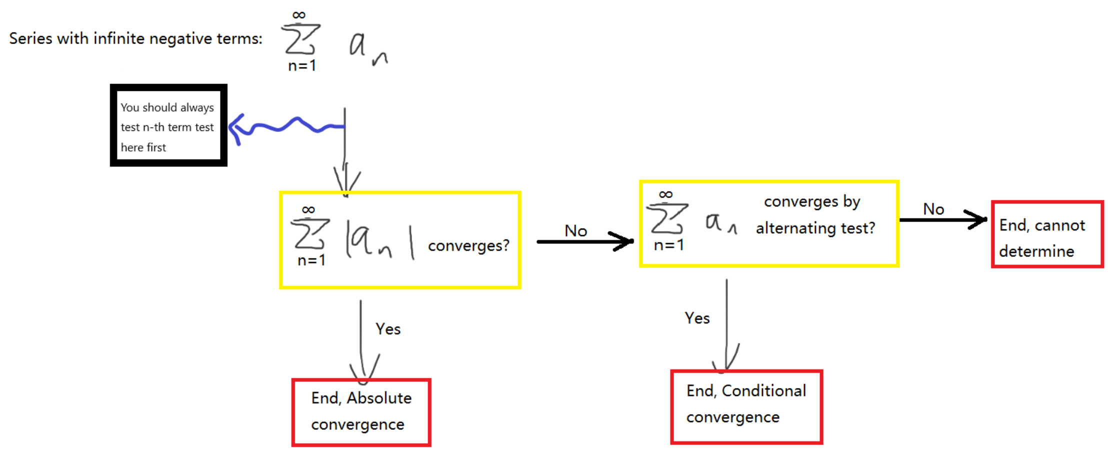
- If $L = 1$, the test does not tell anything about convergence of $\sum_{n=1}^{\infty} a_n$ (!).

7. **Alternating Series Test** A series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1}a_n + \ldots$ converges if $0 < a_{n+1} < a_n$ for all n and $lim_{n\to\infty} a_n = 0$.

   Error of alternating test: let $S = \lim_{n\to\infty} S_n$, then have $|S - S_n| < a_{n+1}$.

Notably, We say that the series $\sum_{n=1}^{\infty} a_n$ is

- absolutely convergent if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ both converge.
- conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

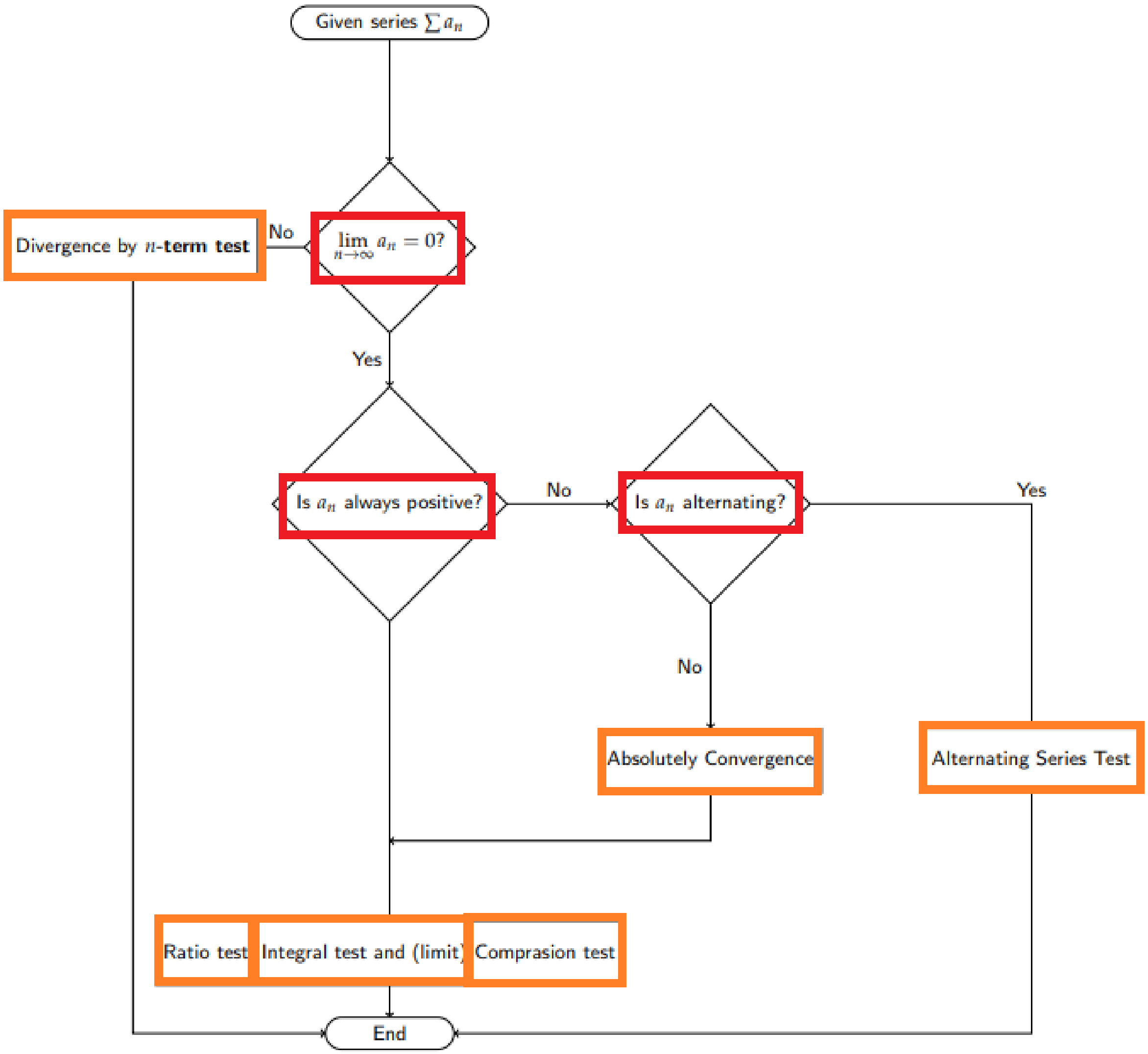


Test we consider for proving convergence:

1. The integral test
2. p-test
3. Comparison test
4. Limit comparison test
5. Check the absolute convergence of the series
6. Ratio Test
7. Alternating Series Test

Test we consider for proving divergence:

1. The integral test
2. p-test
3. Comparison test
4. Limit comparison test
5. Ratio Test
6. Check $\lim_{n\to\infty} \neq 0$ or $\lim_{n\to\infty}$ does not exist.



6.3 Geometric Series

There is a special series that we learn about, which is the Geometric Series, notice that the formula on the right hand side is what we called closed form. A finite geometric series has the form

$$a + ax + ax^2 + \cdots + ax^{n2} + ax^{n1} = \frac{a(1 - x^n)}{1 - x} \text{ For } x \neq 1$$

An infinite geometric series has the form

$$a + ax + ax^2 + \cdots + ax^{n2} + ax^{n1} + ax^n + \cdots = \frac{a}{1 - x} \text{ For } |x| < 1$$

6.4 Power Series

Definition 6.1. A power series about $x = a$ is a sum of constants times powers of $(x - a)$:
$C_0 + C_1(x-a) + C_2(x-a)^2 + \ldots + C_n(x-a)^n + \ldots = \sum_{n=0}^{\infty} C_n(x-a)^n$.

Moreover, each power series falls into one of the three following cases, characterized by its radius of convergence, R.

- The series converges only for $x = a$; the radius of convergence is defined to be $R = 0$.
- The series converges for all values of x; the radius of convergence is defined to be $R = \infty$.
- There is a positive number R, called the radius of convergence, such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$.

How to find radius of convergence: consider ratio test
The interval of convergence is the interval between $a - R$ and $a + R$, including any endpoint where the series converges.

6.5 Taylor Polynomial

Taylor Polynomial of Degree n Approximating $f(x)$ for x near a is

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = a$, or the Taylor poly nomial about $x = a$.

6.6 Taylor Series

Taylor Series for $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = a$, or the Taylor poly nomial about $x = a$.
$f^{(n)}(a) = \{\text{coefficient of } x^n\} * n!$.
Moreover, there are several important cases that we consider, each of them is an Taylor expansion of a function about $x = 0$:

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$ converges for all x

- $\sin(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \cdot (-1)^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ converges for all x

- $\cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \cdot (-1)^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ converges for all x

- $(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k = \sum_{k=0}^{\infty} \frac{p!}{k!(p-k)!} x^k = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$ converges for $-1 < x < 1$.

- $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$,

Moreover, we can definitely find Taylor Series based on the existing series using four methods:
Substitude/Differentiate/Integrate /Multiply

7 Parametric Equations and Polar Coordinate

7.1 Parametric Equations

Summarize, we have the slope: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and the concavity of the parametrized curve to be $\frac{d^2y}{dx^2} = \frac{(dy/dx)/dt}{dx/dt}$
The quantity $v_x = dx/dt$ is the instantaneous velocity in the x-direction; $v_y = dy/dt$ is the instantaneous velocity in the y-direction. And we call that (v_x, v_y) to be the velocity vector.
The instantaneous speed : $v = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(v_x)^2 + (v_y)^2}$.

Moreover, the distance traveled from time a to b is $\int_a^b v(t)dt = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$

7.2 Polar Coordinate

7.2.1 Relation between Cartesian and Polar

Cartesian to Polar: $(x, y) \to (r = \sqrt{x^2 + y^2}, \theta)$ (Here we have that $\tan\theta = \frac{y}{x}$) θ does not have to be $\arctan(\frac{y}{x})$!
Polar to Cartesian: $(r, \theta) \to (x = r\cos\theta, y = r\sin\theta)$

7.2.2 Slope, Arc length and Area in Polar Coordinates

slope of to be $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$
The arc length from angle a to b is $\int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}d\theta = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}d\theta$
Fact: area of the sector is $1/2r^2\theta$, we have that for a curve $r = f(\theta)$, with $f(\theta)$ continuously of the same sign, the area of the region enclosed is $\frac{1}{2}\int_a^b f(\theta)^2 d\theta$