Math 116 — Practice for Exam 2

Generated November 5, 2018

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 14 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2014	3	8		12	
Fall 2015	3	13		10	
Fall 2014	3	6		8	
Fall 2017	2	8		9	
Winter 2016	2	6		15	
Winter 2012	3	4		11	
Winter 2013	2	4		13	
Fall 2013	2	6		12	
Winter 2014	2	10		12	
Fall 2014	2	7		8	
Winter 2015	2	10		15	
Fall 2016	2	9		10	
Winter 2017	2	5		10	
Winter 2018	2	11		10	
Total				155	

Recommended time (based on points): 152 minutes

8. [12 points] Suppose a_n and b_n are sequences of positive numbers with the following properties.

- $\sum_{n=1}^{\infty} a_n$ converges.
- $\sum_{n=1}^{\infty} b_n$ diverges.
- $0 < b_n \le M$ for some positive number M.

For each of the following questions, circle the correct answer. No justification is necessary.

a. [2 points] Does the series $\sum_{n=1}^{\infty} a_n b_n$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

b. [2 points] Does the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

c. [2 points] Does the series $\sum_{n=1}^{\infty} \sqrt{b_n}$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

d. [2 points] Does the series $\sum_{n=1}^{\infty} \sin(a_n)$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

e. [2 points] Does the series $\sum_{n=1}^{\infty} (a_n + b_n)^2$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

f. [2 points] Does the series $\sum_{n=1}^{\infty} e^{-b_n}$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

13. [10 points] Suppose a_n and b_n are sequences with the following properties.

- $\sum_{n=1}^{\infty} a_n$ converges.
- $n \le b_n \le e^n$.

For each of the following statements, decide whether the statement is always true, sometimes true, or never true. Circle your answer. No justification is necessary. **You only need to answer 5 of the 7 questions**. Only answer the 5 questions you want graded. If it is unclear which 5 questions are being answered, the first 5 questions you answer will be graded.

a. [2 points] The sequence $\frac{1}{b_n}$ diverges.

ALWAYS **SOMETIMES** $\boxed{\textbf{NEVER}}$

b. [2 points] The sequence a_n is bounded.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

c. [2 points] The series $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

d. [2 points] The series $\sum_{n=1}^{\infty} e^{-a_n}$ converges.

ALWAYS **SOMETIMES** $\boxed{\textbf{NEVER}}$

e. [2 points] The series $\sum_{n=1}^{\infty} a_n^2$ diverges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

f. [2 points] The series $\sum_{n=1}^{\infty} a_n b_n$ converges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

g. [2 points] The series $\sum_{n=1}^{\infty} \frac{b_n}{n!}$ converges.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

6. [8 points] Suppose that $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are all positive, differentiable functions. Suppose that

$$0 < f(x) < \frac{1}{x} < g(x) < \frac{1}{x^2}$$

for all $0 < x < 1$, and that

$$0 < h(x) < \frac{1}{x^2} < k(x) < \frac{1}{x}$$

for $x > 1$. Determine whether the following statements are always, sometimes or never true by circling the appropriate answer. No justification is necessary.

a. [2 points] $\displaystyle\int_0^1 g(x)dx$ converges.

Always **Sometimes** $\boxed{\textbf{Never}}$

b. [2 points] $\displaystyle\int_0^1 f(x)dx$ diverges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

c. [2 points] $\displaystyle\sum_{n=1}^{\infty} h(n)$ diverges.

Always **Sometimes** $\boxed{\textbf{Never}}$

d. [2 points] $\displaystyle\sum_{n=1}^{\infty} k(n)$ converges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

8. [9 points] For each of parts **a** through **c** below, circle all of the statements that must be true.
Circle "NONE OF THESE" if none of the statements must be true.
You must circle at least one choice to receive any credit.
No credit will be awarded for unclear markings. No justification is necessary.

a. [3 points] Suppose $f(x)$ is a continuous and decreasing function on the interval $[0,2]$ with $f(0) = 1$ and $f(2) = 0$.

Let a be a constant with $0 < a < 1$. Consider the integral $\displaystyle\int_a^2 \frac{1}{f(x)}\,dx$.

i. This integral is not improper.

ii. This integral converges by direct comparison with the constant function 1.

iii. This integral converges by direct comparison with the function $f(x)$.

iv. This integral converges for some values of a between 0 and 1 but diverges for other values of a between 0 and 1.

v. $\boxed{\text{NONE OF THESE}}$

b. [3 points] Suppose $g(x)$ is a positive and decreasing function that is defined and continuous on the open interval $(5,\infty)$ such that

$\displaystyle\int_{10}^{\infty} g(x)\,dx$ converges and $\displaystyle\int_5^8 g(x)\,dx$ diverges.

i. $\boxed{\text{The series } \displaystyle\sum_{n=20}^{\infty} g(n) \text{ converges.}}$

ii. $\boxed{\text{The series } \displaystyle\sum_{n=12}^{\infty} \frac{1}{g(n)} \text{ diverges.}}$

iii. $\boxed{\text{The sequence } c_n = \displaystyle\int_{15}^{n} g(x)\,dx,\ n \geq 15, \text{ converges.}}$

iv. $\boxed{\text{The integral } \displaystyle\int_5^7 g(x)\,dx \text{ diverges.}}$

v. NONE OF THESE

c. [3 points] Consider the sequence $a_n = \dfrac{1}{\ln(n)}$, $n \geq 2$.

Note: Due to a typo on the original exam (corrected here), all students received full credit for part **c**.

i. $\boxed{\displaystyle\lim_{n\to\infty} a_n = 0.}$

ii. The series $\displaystyle\sum_{n=2}^{\infty} a_n$ converges.

iii. $\boxed{\text{The series } \displaystyle\sum_{n=2}^{\infty} a_n \text{ diverges.}}$

iv. $\boxed{\text{The series } \displaystyle\sum_{n=2}^{\infty} (-1)^n a_n \text{ converges.}}$

v. NONE OF THESE

6. [15 points] In the following questions, circle the correct answer. You do not need to show any work, but make sure your answer is clear. No points will be given for unclear answers.

a. [3 points] The value of A for which the function $y = e^{x^2+A^3x}$ solves the equation $y' + 8y = 2xy$ is

$0 \qquad \boxed{-2} \qquad -8 \qquad -\sqrt{8} \qquad 1$

b. [3 points] The function g is positive, decreasing and differentiable. The solution curves of the differential equation $y' = e^{-x}g(y)$ are

concave up $\qquad$ $\boxed{\text{concave down}}$ $\qquad$ changing concavity

c. [3 points] Suppose that $h(x)$ is an increasing differentiable function with $h(0) = 0$ and $\lim_{x\to\infty} h(x) = 5$. The value of the integral $\int_0^\infty (h(x))^4 h'(x)\,dx$

diverges $\qquad$ $\boxed{\text{is } 5^4}$ $\qquad$ is $5^4 - \frac{1}{5}$ $\qquad$ is 1 $\qquad$ is 0

d. [3 points] Suppose $a \geq 1$ is a constant, and the function h satisfies $\frac{1}{x^{1/a}} \leq h(x) \leq \frac{1}{x^a}$ for $0 \leq x \leq 1$. The integral $\int_0^1 (h(x))^2 dx$ converges

always $\qquad$ never $\qquad$ $\boxed{\text{sometimes}}$

e. [3 points] The function f satisfies $\frac{1}{x^3} \leq f(x) \leq \frac{1}{x}$ for $x \geq 1$ and $f(x) = g(x^2)$. The integral $\int_1^\infty \frac{g(x)}{x} dx$ converges

$\boxed{\text{always}}$ $\qquad$ never $\qquad$ sometimes

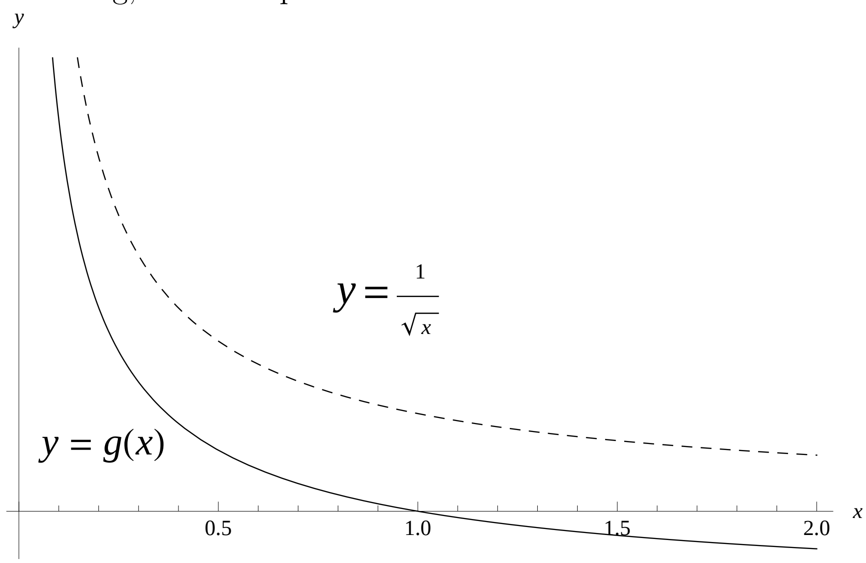
4. [11 points]

a. [2 points] Let $g(x)$ be a continuous function for $x > 0$ and let $G(x)$ be the antiderivative of $g(x)$ with $G(1) = 0$. Write a formula for $G(x)$.

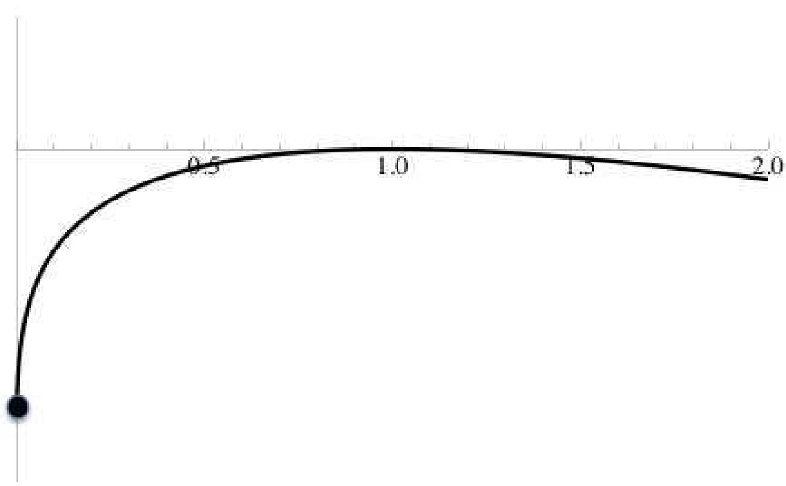
Solution: $G(x) = \int_1^x g(t)dt$

b. [5 points] The graph of $g(x)$ is shown below. The function $g(x)$ has a vertical asymptote at $x = 0$ and $g(x) < \dfrac{1}{\sqrt{x}}$ for $x > 0$.

Sketch the graph of $G(x)$ for $0 \le x \le 2$. Make sure you indicate where $G(x)$ has asymptotes, local maxima, or local minima, as well as where $G(x)$ is increasing, decreasing, concave up or concave down.



Solution:



c. [4 points] Suppose $h(x)$ and $f(x)$ are continuous functions satisfying

i. $0 < f(x) \leq \dfrac{1}{x^p}$ for $0 < x \leq 1$.

ii. $\dfrac{1}{x^{p+\frac{1}{2}}} \leq h(x) \leq \dfrac{1}{x^p}$ for $x \geq 1$.

Decide whether each of the following expressions converge, diverge or if there is not enough information available to conclude.

Solution:

i. If $p = \dfrac{1}{2}$,

(a) $\lim\limits_{x\to\infty} h(x)$

Converges Diverges Not possible to conclude.

(b) $\displaystyle\int_1^\infty h(x)dx$:

Converges **Diverges** Not possible to conclude.

ii. If $p = 2$,

(a) $\displaystyle\int_1^\infty h(x)dx$:

Converges Diverges Not possible to conclude.

(b) $\displaystyle\int_0^1 f(x)dx$

Converges Diverges **Not possible to conclude.**

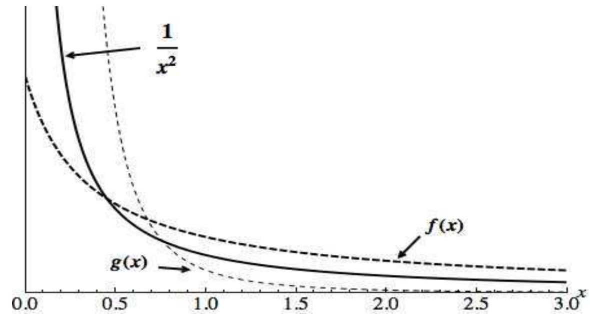
4. [13 points]

a. [8 points] Consider the functions $f(x)$ and $g(x)$ where

$$\frac{1}{x^2} \le g(x) \quad \text{for} \quad 0 < x < \frac{1}{2}.$$

$$g(x) \le \frac{1}{x^2} \quad \text{for} \quad 1 < x$$

$$\frac{1}{x^2} \le f(x) \quad \text{for} \quad 1 < x.$$



Using the information about $f(x)$ and $g(x)$ provided above, determine which of the following integrals is convergent or divergent. Circle your answers. If there is not enough information given to determine the convergence or divergence of the integral circle NI.

i) $\displaystyle\int_1^\infty f(x)dx$ CONVERGENT DIVERGENT **NI**

ii) $\displaystyle\int_1^\infty g(x)dx$ **CONVERGENT** DIVERGENT NI

iii) $\displaystyle\int_0^1 f(x)dx$ **CONVERGENT** DIVERGENT NI

iv) $\displaystyle\int_0^1 g(x)dx$ CONVERGENT **DIVERGENT** NI

b. [5 points] Does $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}dx$ converge or diverge? If the integral converges, compute its value. Show all your work. Use u substitution.

Solution:

$$\int_e^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}dx$$

$$\text{using } u = \ln x \qquad = \lim_{b\to\infty}\int_1^{\ln b} \frac{1}{u^2}dx = \lim_{b\to\infty} -\frac{1}{u}\Big|_1^b = \lim_{b\to\infty} 1 - \frac{1}{\ln b} = 1 \qquad \text{converges.}$$

6. [12 points] Determine the convergence or divergence of the following improper integrals. Justify your answers. Make sure to properly cite any results of convergence or divergence of integrals that you use. If you use the comparison test, be sure to show all your work. Circle your answer.

a. [4 points] $\displaystyle\int_3^\infty \frac{1}{\sqrt[3]{x}+e^{2x}}dx.$ **CONVERGES** DIVERGES

Solution: Since

$$\frac{1}{\sqrt[3]{x}+e^{2x}} \le \frac{1}{e^{2x}} = e^{-2x}$$

and $\displaystyle\int_3^\infty e^{-2x}dx$ converges then $\displaystyle\int_3^\infty \frac{1}{\sqrt[3]{x}+e^{2x}}dx$ converges.

b. [4 points] $\displaystyle\int_2^\infty \frac{3+b\sin^2(x^4)}{x^5}dx$, where b is a positive constant.

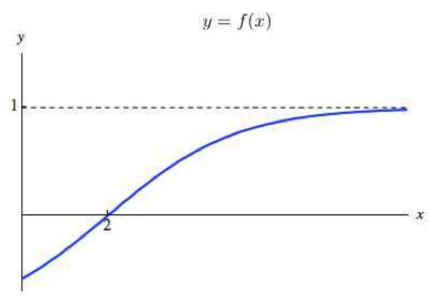
CONVERGES DIVERGES

Solution: Since

$$\frac{3+b\sin^2(x^4)}{x^5} \le (3+b)\left(\frac{1}{x^5}\right)$$

and $\displaystyle\int_2^\infty \frac{1}{x^5}dx$ converges ($p>1$) then $\displaystyle\int_2^\infty \frac{3+b\sin^2(x^4)}{x^5}dx$ converges.

c. [4 points] Let $f(x)$ be the differentiable function shown below. Note that $f(x)$ has a horizontal asymptote at $y=1$.



Does $\displaystyle\int_2^\infty \frac{f'(x)}{1+f(x)}dx$ converge or diverge? Circle your answer. If it converges, find its value.

CONVERGES DIVERGES

Solution:

$$\int_2^\infty \frac{f'(x)}{1+f(x)}dx = \lim_{b\to\infty}\int_2^b \frac{f'(x)}{1+f(x)}dx$$

$$= \lim_{b\to\infty} \ln|1+f(x)|\Big|_2^b = \lim_{b\to\infty} \ln|1+f(b)| - \ln|1+f(2)| = \ln 2.$$

10. [12 points] Suppose that $g(x)$ and $h(x)$ are positive continuous functions on the interval $(0, \infty)$ with the following properties:

- $\int_1^\infty g(x)\,dx$ converges.
- $\int_0^1 g(x)\,dx$ diverges.
- $e^{-x} \leq h(x) \leq \frac{1}{x}$ for all x in $(0, \infty)$.

For each of the following questions, circle the correct answer.

a. [2 points] Does the integral $\displaystyle\int_1^\infty h(x)^2\,dx$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

b. [2 points] Does the integral $\displaystyle\int_0^1 h(x)\,dx$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

c. [2 points] Does the integral $\displaystyle\int_1^\infty h(1/x)\,dx$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

d. [2 points] Does the integral $\displaystyle\int_0^1 g(x)h(x)\,dx$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

e. [2 points] Does the integral $\displaystyle\int_1^\infty g(x)h(x)\,dx$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

f. [2 points] Does the integral $\displaystyle\int_1^\infty e^x g(e^x)\,dx$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

7. [8 points] Suppose that $f(x)$ is a differentiable function, defined for $x > 0$, which satisfies the inequalities $0 \leq f(x) \leq \frac{1}{x}$ for $x > 0$. Determine whether the following statements are always, sometimes or never true by circling the appropriate answer. No justification is necessary.

a. [2 points] $\int_1^\infty f(x)dx$ converges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

b. [2 points] $\int_1^\infty (f(x))^2 dx$ converges.

$\boxed{\textbf{Always}}$ **Sometimes** **Never**

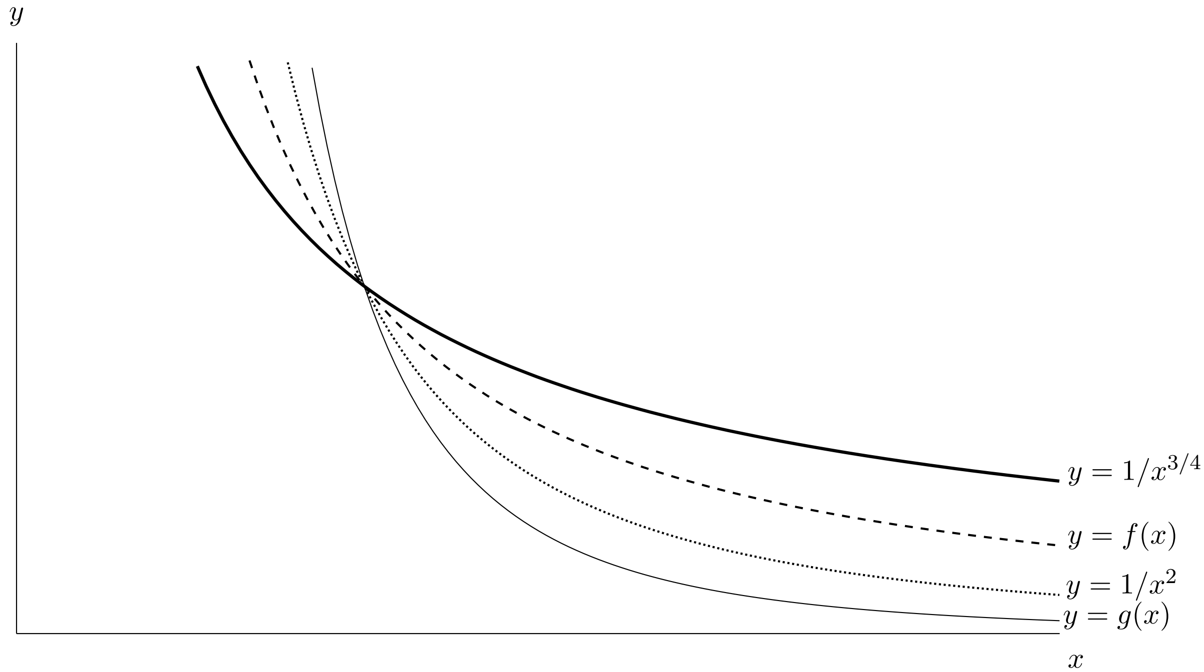
c. [2 points] $\int_0^1 f(x)dx$ converges.

Always $\boxed{\textbf{Sometimes}}$ **Never**

d. [2 points] $\int_1^\infty e^{f(x)} dx$ converges.

Always **Sometimes** $\boxed{\textbf{Never}}$

10. [15 points] Consider the graph below depicting four functions for $x > 0$. The only point of intersection between any two of the functions is at $x = 1$. The functions $f(x)$ and $g(x)$ are both differentiable, and they each have $y = 0$ as a horizontal asymptote and $x = 0$ as a vertical asymptote.



Use the graph to determine whether the following quantities converge or diverge, and circle the appropriate answer. If there is not enough information to determine convergence or divergence, circle "not enough information". You do not need to show your work.

a. [3 points] $\displaystyle\int_1^\infty f(x)\,dx$

Solution:

CONVERGES DIVERGES $\boxed{\text{NOT ENOUGH INFORMATION}}$

b. [3 points] $\displaystyle\int_0^1 g(x)\,dx$

Solution:

CONVERGES $\boxed{\text{DIVERGES}}$ NOT ENOUGH INFORMATION

c. [3 points] $\displaystyle\int_0^1 g'(x)\,e^{-g(x)}\,dx$

Solution:

$\boxed{\text{CONVERGES}}$ DIVERGES NOT ENOUGH INFORMATION

d. [3 points] $\displaystyle\int_1^\infty \sqrt{g(x)}\,dx$

Solution:

CONVERGES DIVERGES $\boxed{\text{NOT ENOUGH INFORMATION}}$

e. [3 points] The volume of the solid formed by rotating the region between $f(x)$ and the x-axis from $x = 1$ to $x = \infty$ about the x-axis

Solution:

$\boxed{\text{CONVERGES}}$ DIVERGES NOT ENOUGH INFORMATION

9. [10 points] Suppose that f is function with the following properties:

f is differentiable $\qquad$ $f(x) > 0$ for all x $\qquad$ $\int_1^\infty f(x)\,dx$ converges.

For each of the following parts, determine whether the statement is always, sometimes, or never true by circling the appropriate answer. No justification is needed.

a. [2 points] $\int_{500}^\infty 1000f(x)\,dx$ converges.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

b. [2 points] $\int_1^\infty (f(x))^{2/3}\,dx$ converges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

c. [2 points] $\int_1^\infty (f(x))^{3/2}\,dx$ converges.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

d. [2 points] $\int_0^1 f\left(\frac{1}{x}\right)\,dx$ converges.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

e. [2 points] $\int_1^\infty \frac{f'(x)}{f(x)}\,dx$ converges. $\left(\text{Note: } \frac{f'(x)}{f(x)} = \frac{d}{dx}\ln(f(x)).\right)$

ALWAYS **SOMETIMES** $\boxed{\textbf{NEVER}}$

5. [10 points] Let $f(x)$ and $g(x)$ be two functions that are differentiable on $(0,\infty)$ with continuous derivatives and which satisfy the following inequalities for all $x \geq 1$:

$$\frac{1}{x} \leq f(x) \leq \frac{1}{x^{1/2}} \qquad \text{and} \qquad \frac{1}{x^2} \leq g(x) \leq \frac{1}{x^{3/4}}.$$

For each of the following, determine whether the integral always, sometimes, or never converges. Indicate your answer by circling the one word that correctly fills the answer blank. No justification is necessary. No credit will be awarded for unclear markings.

a. [2 points] $\displaystyle\int_1^\infty \sqrt{f(x)}\,dx$ __________ converges.

Always Sometimes $\boxed{\text{Never}}$

b. [2 points] $\displaystyle\int_3^\infty 4000g(x)\,dx$ __________ converges.

Always $\boxed{\text{Sometimes}}$ Never

c. [2 points] $\displaystyle\int_1^\infty f(x)g(x)\,dx$ __________ converges.

$\boxed{\text{Always}}$ Sometimes Never

d. [2 points] $\displaystyle\int_5^\infty g'(x)e^{g(x)}\,dx$ __________ converges.

$\boxed{\text{Always}}$ Sometimes Never

e. [2 points] $\displaystyle\int_1^\infty f'(x)\ln(f(x))\,dx$ __________ converges.

$\boxed{\text{Always}}$ Sometimes Never

11. [10 points] You work for a temp agency. Today you fill in for Russ Weterson, doing important work for the city. On Mr. Weterson's desk you find the following problems with a note: "Russ, the Mayor needs these problems done yesterday. -Brontel"

Suppose $f(x)$ and $g(x)$ are positive, continuous, decreasing functions such that

1. $\int_1^\infty f(x)\,dx$ converges, and
2. $0 \le g(x) \le 9$ for all real numbers x.

Determine whether the following expressions must converge, must diverge, or whether convergence cannot be determined. **No justification required.**

a. [2 points] $\displaystyle\int_1^\infty \frac{1}{f(x)}\,dx$

CONVERGES $\boxed{\text{DIVERGES}}$ CANNOT BE DETERMINED

b. [2 points] $\displaystyle\sum_{n=1}^{\infty} f(n)$

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT BE DETERMINED

c. [2 points] $\displaystyle\int_1^\infty f(x)g(x)\,dx$

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT BE DETERMINED

d. [2 points] $\displaystyle\sum_{n=1}^{\infty} f(n)^{g(n)}$

CONVERGES DIVERGES $\boxed{\text{CANNOT BE DETERMINED}}$

e. [2 points] $\displaystyle\int_1^\infty g(x)\,dx$

CONVERGES DIVERGES $\boxed{\text{CANNOT BE DETERMINED}}$