Exam 2 Summary

1 L'Hopital's rule

LHopitals rule: If f and g are differentiable and (below a can be $\pm\infty$)
i)$f(a) = g(a) = 0$ for finite a,
Or ii)$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$,
Or iii)$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

1.1 Dominance

We say that g dominates f as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.

1.2 How to determine some bad limit?

There are several types of the limits that is "bad" which requires L'Hopital's rule to calculate: $0/0, \infty/\infty, \infty \cdot 0$. Although the first two cases we can use L'Hopital's rule to calculate, the others we cannot use it directly.
Read the book, and there are several things that we can consider.

- Consider taking log.
- Consider $1/f(x)$ so that we can transform ∞ to '1/0' or 0 to '$1/\infty$'.

2 Improper integral

Formal definition of the improper integral I will let you read the book carefully, they are in the box. However, informally, there are two types of improper integral which we just interpret them as a limit.

- The first case is where we have the limit of the integration goes to infinity, i.e. $\lim_{b\to\infty} \int_a^b f(x)dx$.
- The integrand goes to infinity as $x \to a$.

2.1 Converges or diverges?

The basic question that one want to know about the improper integral is basically is it well defined?
This turns to ask if an improper integral converges or not.
There are four ways people ususally use to check this fact.

1. Check by definition, this means check the limit directly.

2. p-test.

	$p<1$	$p=1$	$p>1$
Type I : $\int_a^\infty \frac{1}{x^p}\,dx$	diverges	$= \ln x \big\vert_a^\infty \Rightarrow$ diverges	converges
Type II : $\int_0^a \frac{1}{x^p}\,dx$	converges	$= \ln x \big\vert_0^a \Rightarrow$ diverges	diverges

3. Exponential decay test.

$$\int_0^\infty e^{-ax}dx$$

converges for $a>0$.

4. Comparison test.
If $f(x) \geq g(x) \geq 0$ on the interval $[a,\infty]$ then,

   - If $\int_a^\infty f(x)dx$ converges then so does $\int_a^\infty g(x)dx$.
   - If $\int_a^\infty g(x)dx$ diverges then so does $\int_a^\infty f(x)dx$.

5. Limit Comparison theorem.
Limit Comparison Test. If $f(x)$ and $g(x)$ are both positive on the interval $[a,b)$ where b could be a real number or infinity. and

$$\lim_{x\to b}\frac{f(x)}{g(x)} = C$$

such that $0<C<\infty$ then the improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are either both convergent or both divergent.

3 Probability

3.1 PDF and CDF

Definition 3.1. A function $p(x)$ is a **probability density function** or PDF if it satisfies the following conditions

- $p(x) \geq 0$ for all x.
- $\int_{-\infty}^{\infty} p(x) = 1$.

Definition 3.2. A function $P(t)$ is a **Cumulative Distribution Function** or cdf, of a density function $p(t)$, is defined by

$$P(t) = \int_{-\infty}^{t} p(x)dx$$

Which means that $P(t)$ is the antiderivative of $p(t)$ with the following properties:

- $P(t)$ is increasing and $0 \leq P(t) \leq 1$ for all t.
- $\lim_{t\to\infty} P(t) = 1$.
- $\lim_{t\to-\infty} P(t) = 0$.

Moreover, we have $\int_a^b p(x)dx = P(b) - P(a)$.

3.2 Probability, mean and median

Probability

Let us denote X to be the quantity of outcome that we care (X is in fact, called the random variable).

$$\mathbb{P}\{a \leq X \leq b\} = \int_a^b p(x)dx = P(b) - P(a)$$
$$\mathbb{P}\{X \leq t\} = \int_{-\infty}^{t} p(x)dx = P(t)$$
$$\mathbb{P}\{X \geq s\} = \int_{s}^{\infty} p(x)dx = 1 - P(s)$$

The mean and median

Definition 3.3. A **median** of a quantity X is a value T such that the probability of $X \leq T$ is 1/2. Thus we have T is defined by the value such that

$$\int_{-\infty}^{T} p(x)dx = 1/2$$

or

$$P(T) = 1/2$$

.

Definition 3.4. A **mean** of a quantity X is the value given by

$$Mean = \frac{\text{Probability of all possible quantity}}{\text{Total probability}} = \frac{\int_{-\infty}^{\infty} xp(x)dx}{\int_{-\infty}^{\infty} p(x)dx} = \frac{\int_{-\infty}^{\infty} xp(x)dx}{1} = \int_{-\infty}^{\infty} xp(x)dx.$$

Normal Distribution

Definition 3.5. A normal distribution has a density function of the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean of the distribution and σ is the standard deviation, with $\sigma > 0$. The case $\mu = 0$, $\sigma = 1$ is called the standard normal distribution.

4 Sequences and Series

4.1 Sequence

Definition 4.1. A sequence is an enumerated collection of objects in which repetitions are allowed. We denote the sequence $a_1, a_2, \ldots, a_n \ldots$ as (a_n).

Note that for sequence, there are two things that we will usually concern. The first one is the convergence of the sequence itself, which is defined as

Definition 4.2. The sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ has a limit L, written $\lim_{n\to\infty} s_n = L$, if s_n is as close to L as we please whenever n is sufficiently large. If a limit, L, exists, we say the sequence converges to its limit L. If no limit exists, we say the sequence diverges.

If we think about the situation more clearly, we will see that, in the definition it actually encodes an information: A convergent sequence is bounded. Is the converse true here? Unfortunately, it is not true that a bounded sequence is convergent. However, by the following theorem, we knows when will the bounded sequence becomes convergent.

Theorem 4.1. Bounded Monotone sequence converges: If a sequence s_n is bounded and monotone, it converges.

4.2 Series

There is another thing that we will usually concern.
Consider the partial sum of sequence s_n, i.e., $S_n = \sum_{i=1}^{n} s_i$, then we will see that the partial sum forms a sequence as well. Therefore there is a natural question to ask here, when will the sequence S_n of partial sums converges?

Definition 4.3. The associated series for a sequence (a_n) is defined as the ordered sum $\sum_{n=1}^{\infty} a_n$.

Definition 4.4. If the sequence S_n of partial sums converges to S, so $\lim_{n\to\infty} S_n = S$, then we say the series $\sum_{n=1}^{\infty} a_n$ converges and that its sum is S. We write $\sum_{n=1}^{\infty} a_n = S$. If $\lim_{n\to\infty} S_n$ does not exist, we say that the series diverges.

There are several properties for convergent series, which is super useful, summarized as below.

Theorem 4.2. Convergence Properties of Series

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and if k is a constant, then
   $\sum_{n=1}^{\infty}(a_n + b_n)$ converges to$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
   $\sum_{n=1}^{\infty} ka_n$ converges to $k\sum_{n=1}^{\infty} a_n$

2. Changing a finite number of terms in a series does not change whether or not it converges,

3. If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges. (**Remember this!**)

4. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges if $k \neq 0$.

Moreover, there are several test to determine if a series is convergent, detailed discussion about those is in class.

1. **The Integral Test**
   Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.
   a. If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^\infty a_n$ an converges.
   b. If $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^\infty a_n$ an diverges.

2. **p-test**
   The p-series $\sum_{n=1}^\infty 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.

3. **Comparison Test**
   Suppose $0 \leq a_n \leq b_n$ for all n beyond a certain value.
   a. If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges.
   b. If $\sum_{n=1}^\infty a_n$ diverges, then $\sum_{n=1}^\infty b_n$ diverges.

4. **Limit Comparison Test**
   Suppose $a_n > 0$ and $b_n > 0$ for all n. If $\lim_{n\to\infty} a_n/b_n = c$ where $c > 0$, then the two series $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ either both converge or both diverge.

5. **Convergence of Absolute Values Implies Convergence**
   If $\sum_{n=1}^\infty |a_n|$ converges, then so does $\sum_{n=1}^\infty a_n$.

6. **The Ratio Test** For a series $\sum_{n=1}^\infty a_n$, suppose the sequence of ratios $|a_{n+1}|/|a_n|$ has a limit: $\lim_{n\to\infty} |a_{n+1}|/|a_n| = L$, then

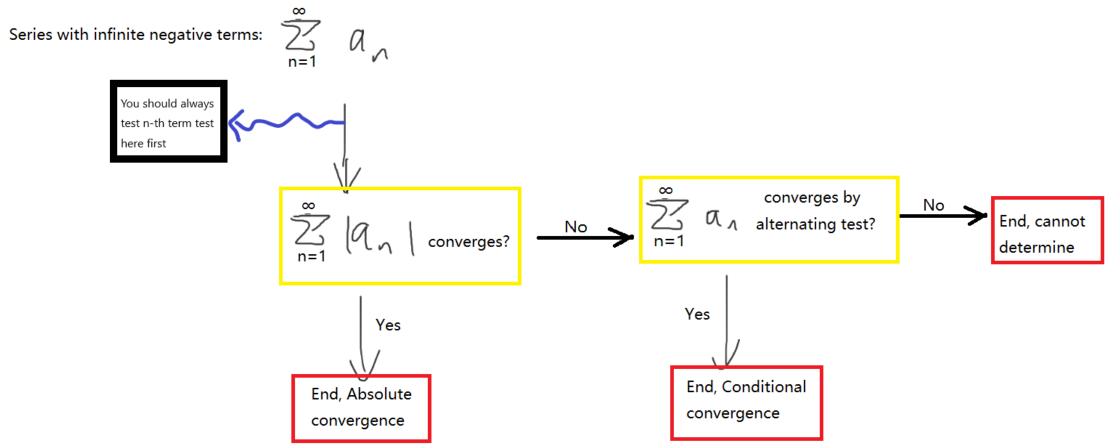
   - If $L < 1$, then $\sum_{n=1}^\infty a_n$ converges.
   - If $L > 1$, or if L is infinite, then $\sum_{n=1}^\infty a_n$ diverges.
   - If $L = 1$, the test does not tell us anything about the convergence of $\sum_{n=1}^\infty a_n$ (**Important!**).

7. **Alternating Series Test** A series of the form $\sum_{n=1}^\infty (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1} a_n + \ldots$ converges if $0 < a_{n+1} < a_n$ for all n and $lim_{n\to\infty} a_n = 0$.

   Error of alternating test: Moreover, let $S = \lim_{n\to\infty} S_n$, then we will have $|S - S_n| < a_{n+1}$.

Notably, We say that the series $\sum_{n=1}^\infty a_n$ is

- absolutely convergent if $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty |a_n|$ both converge.
- conditionally convergent if $\sum_{n=1}^\infty a_n$ converges but $\sum_{n=1}^\infty |a_n|$ diverges.

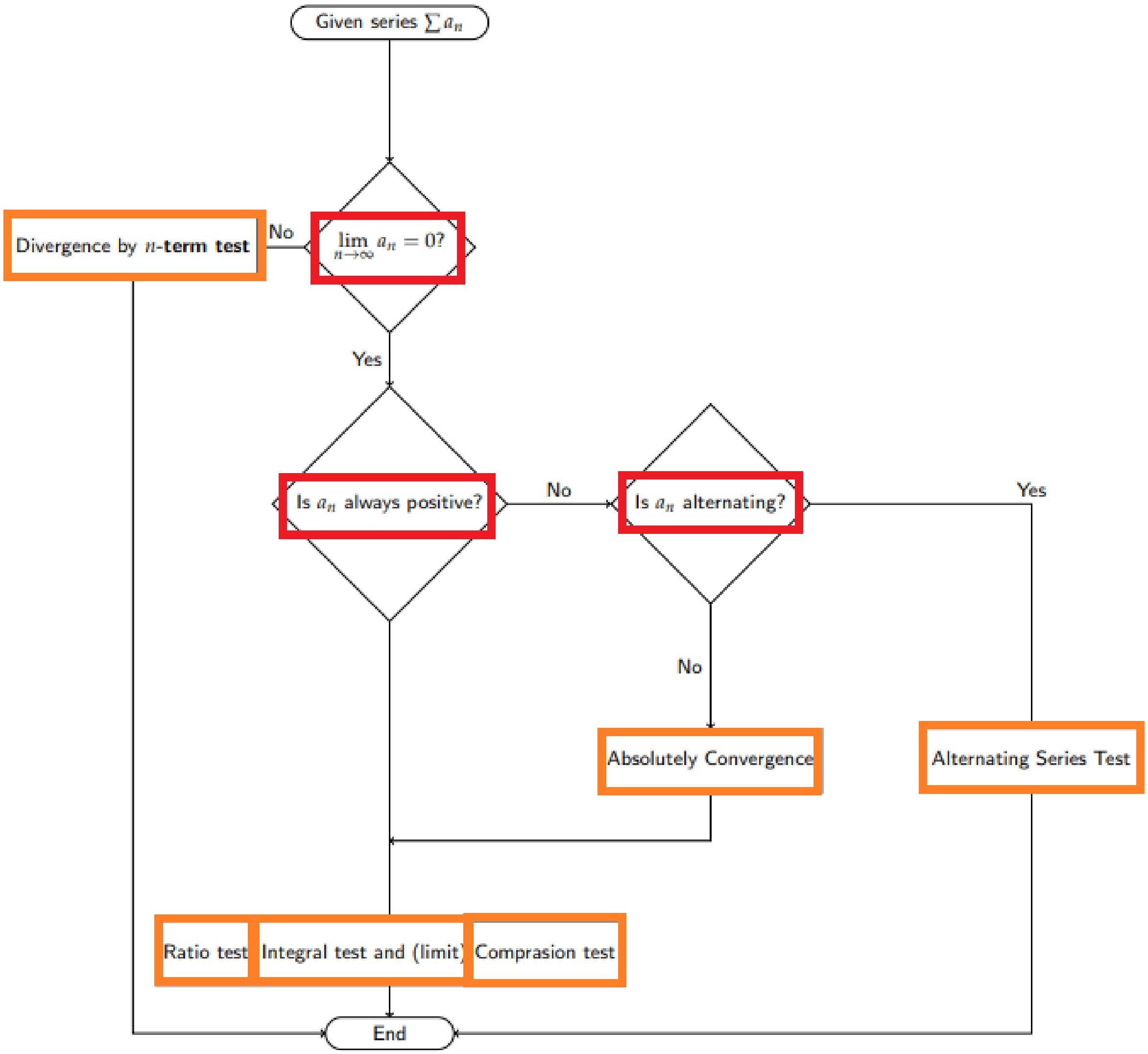


Test we consider for proving convergence:

1. The integral test
2. p-test
3. Comparison test
4. Limit comparison test
5. Check the absolute convergence of the series
6. Ratio Test
7. Alternating Series Test

Test we consider for proving divergence:

1. The integral test
2. p-test
3. Comparison test
4. Limit comparison test
5. Ratio Test
6. Check $\lim_{n\to\infty} \neq 0$ or $\lim_{n\to\infty}$ does not exist.



4.3 Geometric Series

There is a special series that we learn about, which is the Geometric Series, notice that the formula on the right hand side is what we called closed form. A finite geometric series has the form

$$a + ax + ax^2 + \cdots + ax^{n2} + ax^{n1} = \frac{a(1 - x^n)}{1 - x} \text{ For } x \neq 1$$

An infinite geometric series has the form

$$a + ax + ax^2 + \cdots + ax^{n2} + ax^{n1} + ax^n + \cdots = \frac{a}{1 - x} \text{ For } |x| < 1$$

4.4 Power Series

Definition 4.5. A power series about $x = a$ is a sum of constants times powers of $(x - a)$:
$C_0 + C_1(x - a) + C_2(x - a)^2 + \ldots + C_n(x - a)^n + \ldots = \sum_{n=0}^{\infty} C_n(x - a)^n$.

If we fix a specific value of x, we can just consider plugging x with the value we have, and convergence here makes sense.

Definition 4.6. For a fixed value of x, if this sequence of partial sums converges to a limit L, that is, if $\lim_{n\to\infty} S_n(x) = L$, then we say that the power series converges to L for this value of x.

Based on the discussion we will see that, The interval of convergence for a power series is usually centered at a point $x = a$, and extends the same length to both side, thus we denote this length as radius of convergence.

Moreover, each power series falls into one of the three following cases, characterized by its radius of convergence, R.

- The series converges only for $x = a$; the radius of convergence is defined to be $R = 0$.
- The series converges for all values of x; the radius of convergence is defined to be $R = \infty$.
- There is a positive number R, called the radius of convergence, such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$.

The interval of convergence is the interval between $a - R$ and $a + R$, including any endpoint where the series converges.