Intro

- Reminder: please mute your microphone (if you have one) while you're not speaking by clicking the mic button at the bottom. This helps eliminate distracting background noise.
- On the left side of your screen you should see a public chat. Please type a short message like "hi" just to test it out.
- Note that this conference is being recorded, and will be available to all members of the class later.

Intro

1. Review from last time

2. Volumes of Revolution

Review problem

3. [9 points] For each of the following questions, fill in the blank with the letter corresponding to the correct answer from the bottom of the page. No credit will be given for unclear answers.

a. [3 points] The length of the curve $y = \pi e^{-x}$ between $x = 2$ and $x = 4$ is:

____

b. [3 points] Consider the region bounded by the x-axis, $y = -2e^{-2x}$, $x = 2$, and $x = 4$. The volume of the solid whose base is that region and whose cross sections perpendicular to the x-axis are semicircles is:

____

$(A) \int_2^4 \frac{\pi}{2} e^{-4x} dx$

$(B) \int_2^4 \pi(e^{-2x} - 4e^{-4x}) dx$

$(C) \int_2^4 2\pi x(e^{-x} + 2e^{-2x}) dx$

$(D) \int_2^4 \pi\sqrt{1 + e^{-2x}} dx$

$(E) \int_2^4 2\pi e^{-4x} dx$

$(F) \int_2^4 2\pi\sqrt{1 + e^{-4x}} dx$

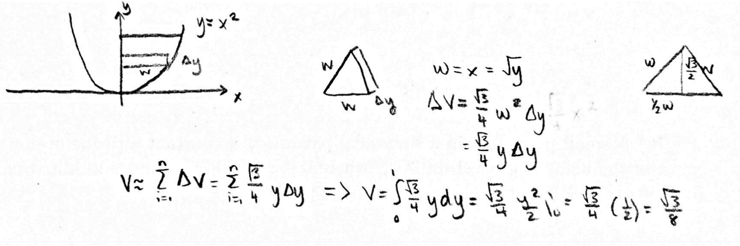
$(G) \int_2^4 \frac{\pi}{2} e^{-2x} dx$

$(H) \int_2^4 \sqrt{1 + \pi^2 e^{-2x}} dx$

$(I) \int_2^4 2\pi(e^{-2x} - 2e^{-4x}) dx$

$(J) \int_2^4 2\pi(e^{-x} + 2e^{-2x}) dx$

Consider the region bounded by $y = x^2$, $y = 1$, and the y-axis, for $x \geq 0$. Find the volume of the solid whose base is the given region and whose cross sections perpendicular to the y-axis are equilateral triangles.



Volumes of Revolution

Consider the region bounded by $y = f(x)$, $x = a$, and $x = b$. When we rotate this over the x-axis, we get a 3D figure:

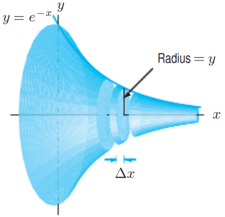


Figure 8.20: A thin strip rotated around the x-axis to form a circular slice

We can approximate the volume by cutting up the figure into disks of thickness Δx.

Disk Method

The i^{th} disk has radius $f(x_i)$ and thickness Δx, so its volume is $\pi(f(x_i))^2\Delta x$. Adding up the volume of each cylinder gives a Riemann sum

$$\sum_{i=1}^{n} \pi(f(x_i))^2\Delta x$$

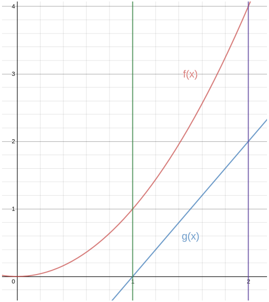
which (letting $n \to \infty$) becomes the integral

$$V = \int_a^b \pi(f(x))^2 dx.$$

This is the **disk method formula**.

Example

A slightly harder example: let $f(x) = x^2$ and $g(x) = -2 + 2x$. Consider the region bounded by $y = f(x)$, $y = g(x)$, $x = 1$, and $x = 2$ as shown below:



Let's find the volume of the solid obtained by rotating this region over the x-axis.

Example (continued)

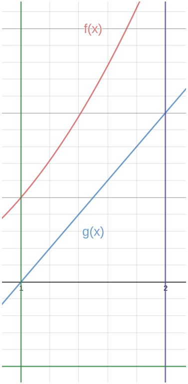
One way to do this: consider the solid obtained by rotating the area under $y = f(x)$ over the x-axis, as well as the solid obtained by rotating the area under $y = g(x)$ over the x-axis. The solid we care about is obtained by hollowing out the second solid from the first one.

So the volume will be

$$V = \int_1^2 \pi(f(x))^2 dx - \int_1^2 \pi(g(x))^2 dx = \int_1^2 \pi((x^2)^2 - (-2 + 2x)^2) dx$$

Example (continued)

What if the region is rotated over the line $y = -1$ rather than $y = 0$?



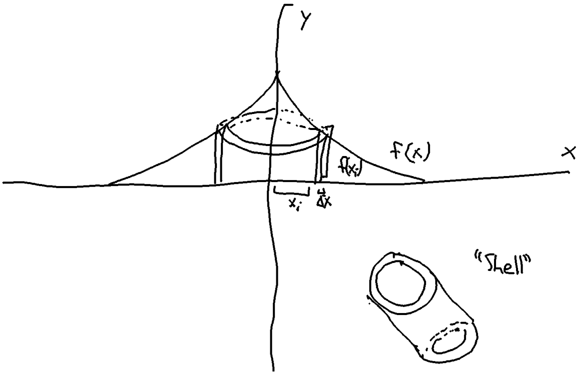
The radius of each disk has now increased by 1. This means the new volume is

$$V = \int_1^2 \pi(f(x)+1)^2 dx - \int_1^2 \pi(g(x)+1)^2 dx$$

Shell Method

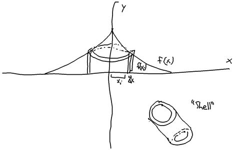
- Let's say we want to rotate a given region (e.g., the area under $y = f(x)$) about the y-axis rather than the x-axis.
- Unlike in the disk method (where we split up our 3D solid into cylindrical disks), we will split up our 3D solid into hollowed-out "shells" and add up their volumes.

Shell Method (continued)



We imagine our solid as consisting of a bunch of shells nested within each other (think of Matryoshka dolls).
What is the volume of the i^{th} shell?

Volume of a Shell



The outer radius is $x_i + \Delta x$ and the inner radius is x_i.

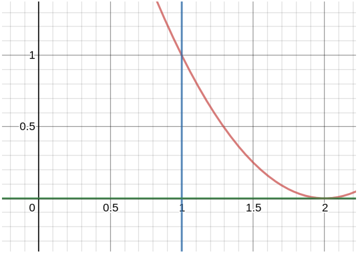
$$\begin{aligned} V &= \pi(x_i + \Delta x)^2 f(x_i) - \pi x_i^2 f(x_i) \\ &= \pi(x_i^2 + 2x_i\Delta x + (\Delta x)^2) f(x_i) - \pi x_i^2 f(x_i) \\ &= 2\pi x_i f(x_i)\Delta x + \pi f(x_i)(\Delta x)^2 \\ &\approx 2\pi x_i f(x_i)\Delta x \end{aligned}$$

So we get a Riemann sum

$$\sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x \text{ which becomes } \int_a^b 2\pi x f(x) dx$$

Example

Consider the region bounded by $y = (x-2)^2$, $x = 1$, and $y = 0$. Let's find the volume of the solid obtained by rotating this region over the y-axis.



Using the shell method formula,

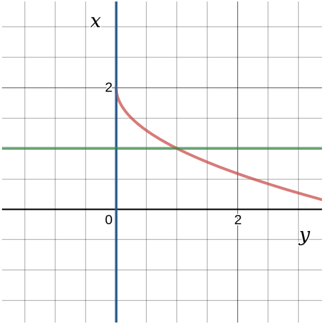
$$V = \int_1^2 2\pi x(x-2)^2 dx = \frac{5\pi}{6}$$

Could we have used the disk method?

Example (continued)

- Remember, the disk method applies when we rotate the area under $y = f(x)$ over the x-axis.
- In this case, we could swap the variables and then rotate over the y-axis.
- Our region is the area between $x = 2 - \sqrt{y}$, $y = 0$, and $x = 1$.

Example (continued)



Notice the labels on the axes. The volume of the solid obtained by rotating over the y-axis will be

$$V = \int_0^1 \pi(2-\sqrt{y})^2 dy - \int_0^1 \pi(1)^2 dy = \frac{11\pi}{6} - \pi = \frac{5\pi}{6}$$

just as before.

Worksheet

These problems concern the region bounded by $y = x^2$, $y = 1$, and the y-axis, for $x \geq 0$. Find the volume of the solid.

a. The solid obtained by rotating the region around the y-axis.
b. The solid obtained by rotating the region about the x-axis.
c. The solid obtained by rotating the region about the line $y = -2$.

Worksheet (continued)

Consider the region bounded by $y = \sqrt{x^2 + 1}$, $y = 0$, $x = 1$, and $x = 2$.

a. Determine the volume of the solid obtained by rotating this region around the y-axis using the **disk** method.
b. Determine the volume of the solid obtained by rotating this region around the y-axis using the **shell** method. (Hint: Find x in terms of y.)