Chapter 5: Definite integral

Definition 0.1. A definite integral of f from a to b is definted as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \textbf{(Limit of Right-hand sum)}$$

or

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x \textbf{(Limit of Left-hand sum)}$$

Here, Left-hand sum and Right-hand sum are equal after taking limits and it the the so-called Riemann Sum.

There are two more type of Riemann sum I would like to discuss in the future, which is the Mid sum and the Trapezoidal sum.
I will only give definition here.

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} f(\frac{x_i + x_{i+1}}{2})\Delta x \textbf{(Mid sum)}$$

and

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}\Delta x \textbf{(Trapezoid sum)}$$

In all these Riemann sum we discussed, we are assuming $\Delta(x) = \dfrac{b-a}{n}$, thus as $n \to \infty$, $\Delta x \to 0$.
Note that

$$\frac{LEFT(n) + RIGHT(n)}{2} = TRAP(n)$$

,

$$MID(n) \neq TRAP(n)$$

.

Remark. Properties of definite integral:

1.
$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

2.
$$\int_b^a f(x)dx + \int_c^b f(x)dx = \int_c^a f(x)dx$$

3.
$$\int_b^a (f(x) \pm g(x))dx = \int_b^a f(x)dx \pm \int_b^a g(x)dx$$

4.
$$\int_b^a cf(x)dx = c\int_b^a f(x)dx$$

5. Symmetry due to the oddity of the function.

Remark. Interpretation of Define Integral as Area under graph of f between $x = a$ and $x = b$, counting positivity.

Important cases discussed on Friday's course:

1. Compute
$$\int_{-1}^{1} \sqrt{1-x^2}dx$$

2. How about
$$\int_{-1}^{1} (\sqrt{1-x^2} - 1)dx?$$

3. Maybe try
$$\int_{-0.5}^{0.5} \tan(x)dx$$

Find out the answer yourself only geometrically, even you know more techniques!
More Importantly there are two major topics I want to mention here and maybe discuss:

- When is the estimation done by Riemann sum a underestimate/overestimate?

  It is also covered in 7.5, check it out and try problem 2.

- Error estimation

  Think about the case where you know that $f(x)$ lies between any pair of $LEFT(n)$ and $RIGHT(n)$, then we see that $|LEFT(n) - f(x)| < |LEFT(n) - RIGHT(n)| = (f(b) - f(a))\Delta x$. This usually gives a bound for n.

Theorem 0.1. The Fundamental Theorem of Calculus is basically the theorem defined below.
If f is continuous on interval $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Application of Definite Integral

1. Average value of function $f(x)$ in $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

.

Here, think about the integration is analogue to summation in the discrete world, then this average value is the analogue of

$$x_{\text{average}} = \frac{x_1 + x_2 + \ldots + x_n}{1 + 1 + 1 + \ldots} = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

where in the integration case is actually

$$x_{\text{average}} = \frac{\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x}{\lim_{n\to\infty}\sum_{i=0}^{n-1} 1 \times \Delta x} = \frac{\int_a^b f(x)dx}{\int_a^b 1dx} = \frac{1}{b-a}\int_a^b f(x)dx$$