Math 116 — Practice for Exam 3

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Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2015	3	4	Bessel	9	
Winter 2012	3	9		10	
Fall 2012	3	3		12	
Winter 2017	3	10		8	
Total				39	

Recommended time (based on points): 47 minutes

4. [9 points] We can define the Bessel function of order one by its Taylor series about $x = 0$, $J_1(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$.

a. [3 points] Compute $J_1^{(2015)}(0)$. Write your answer in exact form and do not try evaluate using a calculator.

Solution: The 2015th derivative of J_1 at $x = 0$ corresponds to 2015! times the 1007th coefficient in the Taylor series above. This gives us that $J_1^{(2015)}(0) = \dfrac{-(2015)!}{(1007)!(1008)!2^{2015}}$.

$$J_1^{(2015)}(0) = \frac{-(2015)!}{(1007)!(1008)!2^{2015}}$$

b. [4 points] Find $P_5(x)$, the Taylor polynomial of degree 5 that approximates $J_1(x)$ near $x = 0$.

Solution: $P_5(x) = \dfrac{x}{2} - \dfrac{x^3}{16} + \dfrac{x^5}{384}$

$$P_5(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}$$

c. [2 points] Use the Taylor polynomial from the previous part to compute $\displaystyle\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3}$.

Solution: $\displaystyle\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \lim_{x\to 0} \frac{-\frac{x^3}{16} + \frac{x^5}{384}}{x^3} = -\frac{1}{16}$

$$\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = -\frac{1}{16}$$

9. [10 points] A second way to approximate the function

$$F(x) = \int_0^x \sqrt{1+9t^4}dt.$$

is by using its Taylor polynomials.

a. [2 points] Find the first three nonzero terms in the Taylor series for the function $\sqrt{1+u}$ about $u = 0$.

Solution:

$$\sqrt{1+u} \approx 1 + \frac{1}{2}u - \frac{1}{8}u^2$$

b. [2 points] Find the first three nonzero terms in the Taylor series for $\sqrt{1+9t^4}$ about $t = 0$.

Solution: Let $u = 9t^4$ then

$$\sqrt{1+9t^4} \approx 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8$$

c. [2 points] Find the first three nonzero terms in the Taylor series for $F(x)$ about $x = 0$.

Solution:

$$F(x) \approx \int_0^x 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8 dt = x + \frac{9}{10}x^5 - \frac{9}{8}x^9$$

d. [2 points] For which values of x do you expect the Taylor series for $F(x)$ about $x = 0$ to converge? Justify your answer.

Solution: We substituted $u = 9t^4$ into the Binomial series. The interval of convergence for the Binomial series is $-1 < u < 1$. Then we expect the series to converge for $0 \leq 9x^4 < 1$. Hence the Taylor series for $F(x)$ about $x = 0$ converges if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

e. [2 points] Use the fifth degree Taylor polynomial for $F(x)$ about $x = 0$ to approximate the value of $F(\frac{1}{2})$.

Solution: $P_5(x) = x + \frac{9}{10}x^5$, then $F\left(\frac{1}{2}\right) \approx P_5\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{9}{10}\left(\frac{1}{2}\right)^5 = 0.528$

3. [12 points] Let

$$I = \int_0^1 \left(1 + \frac{t^2}{2}\right)^{\frac{5}{2}} dt$$

a. [5 points] Approximate the value of I using Right(2) and Mid(2). Write each term in your sums.

Solution:

$$\begin{aligned} \text{Right}(2) &= \frac{1}{2}\left(\left(\frac{9}{8}\right)^{\frac{5}{2}} + \left(\frac{3}{2}\right)^{\frac{5}{2}}\right) \\ &\approx \frac{1}{2}(1.342 + 2.755) \approx 2.04904 \\ \text{Mid}(2) &= \frac{1}{2}\left(\left(\frac{33}{32}\right)^{\frac{5}{2}} + \left(\frac{41}{32}\right)^{\frac{5}{2}}\right) \\ &\approx \frac{1}{2}(1.08 + 1.858) \approx 1.46907 \end{aligned}$$

b. [2 points] Are your estimates of the value of I obtained using Right(2) and Mid(2) guaranteed to be overestimates, underestimates or neither?

Solution: Right = Overestimate (increasing)
Mid = Underestimate (concave up)

c. [3 points] Find the first three nonzero terms of the Taylor series for $g(t) = \left(1 + \frac{t^2}{2}\right)^{\frac{5}{2}}$ about $t = 0$.

Solution: Using the binomial series:

$$\begin{aligned} (1+x)^{\frac{5}{2}} &= 1 + \frac{5}{2}x + \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\frac{x^2}{2!} + \cdots \\ &= 1 + \frac{5}{2}x + \frac{15}{8}x^2 + \cdots \\ \left(1 + \frac{t^2}{2}\right)^{\frac{5}{2}} &= 1 + \frac{5}{2}\left(\frac{t^2}{2}\right) + \frac{15}{8}\left(\frac{t^2}{2}\right)^2 + \cdots \\ &= 1 + \frac{5}{4}t^2 + \frac{15}{32}t^4 + \cdots \end{aligned}$$

d. [2 points] Use your answer from part (c) to estimate I.

Solution:

$$I \approx \int_0^1 1 + \frac{5}{4}t^2 + \frac{15}{32}t^4 dt = t + \frac{5}{12}t^3 + \frac{3}{32}t^5\Big|_{t=0}^{1} = 1 + \frac{5}{12} + \frac{3}{32} = \frac{145}{96} \approx 1.510417$$

10. [8 points] The Taylor series centered at $x = 0$ for a function $F(x)$ converges to $F(x)$ for all x and is given below.

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}$$

a. [3 points] What is the value of $F^{(101)}(0)$?
Make sure your answer is exact. You do not need to simplify.

$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n$ by Taylor's Theorem. So

The x^{101} term is $\frac{F^{(101)}(0)}{101!} x^{101}$. When you plug $n=25$ into the series above you get $-\frac{x^{101}}{50!(101)}$

So $F^{(101)}(0) = \frac{-101!}{50!(101)}$

Answer: $F^{(101)}(0) =$ $-\frac{100!}{50!}$

b. [3 points] Find $P_9(x)$, the 9th degree Taylor polynomial that approximates $F(x)$ near $x = 0$.

$$P_9(x) = \frac{x^{0+1}}{0!(0+1)} - \frac{x^{4+1}}{2!(4+1)} + \frac{x^{8+1}}{4!(8+1)}$$

$$= x - \frac{1}{10}x^5 + \frac{1}{216}x^9$$

c. [2 points] Use your Taylor polynomial from part **b.** to compute

$$\lim_{x \to 0} \frac{F(x) - x}{2x^5}$$

$$\lim_{x \to 0} \frac{F(x) - x}{2x^5} = \lim_{x \to 0} \frac{\left[x - \frac{1}{10}x^5 + \frac{1}{216}x^9\right] - x}{2x^5}$$

$$= \lim_{x \to 0} -\frac{1}{20} + \frac{1}{432}x^4 = -\frac{1}{20}$$