Chapter 10: Taylor Polynomial and Taylor Series

1 Power Series

Definition 1.1. A power series about $x = a$ is a sum of constants times powers of $(x-a)$: $C_0 + C_1(x-a) + C_2(x-a)^2 + \ldots + C_n(x-a)^n + \ldots = \sum_{n=0}^{\infty} C_n(x-a)^n$.

If we fix a specific value of x, we can just consider plugging x with the value we have, and convergence here makes sense.

Definition 1.2. For a fixed value of x, if this sequence of partial sums converges to a limit L, that is, if $\lim_{n\to\infty} S_n(x) = L$, then we say that the power series converges to L for this value of x.

Based on the discussion we will see that, The interval of convergence for a power series is usually centered at a point $x = a$, and extends the same length to both side, thus we denote this length as radius of convergence.

Moreover, each power series falls into one of the three following cases, characterized by its radius of convergence, R.

- The series converges only for $x = a$; the radius of convergence is defined to be $R = 0$.
- The series converges for all values of x; the radius of convergence is defined to be $R = \infty$.
- There is a positive number R, called the radius of convergence, such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

The interval of convergence is the interval between $a - R$ and $a + R$, including any endpoint where the series converges.

Then there is a question arises, how to find this radius of convergence then?

This question can be determined by considering using ratio test on the series, assuming $x \neq a$. The details are included in Chapter 9.5 in the book.

2 Taylor Polynomial and Taylor Series

2.1 Taylor Polynomial

If we try to approximate the function locally using a polynomial, there is one thing we want to acquire, i.e. we want the polynomial $P(x)$ with the property that $P^{(n)}(a) = f^{(n)}(a)$ if we approximate the function at the point $x = a$. Considering merely the situation about $x = 0$, recall what we did in the class, we will have the following.
Taylor Polynomial of Degree n Approximating $f(x)$ for x near 0 is

$$f(x) \approx P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = 0$, or the Taylor poly nomial about $x = 0$.

More generally, Taylor Polynomial of Degree n Approximating $f(x)$ for x near a is

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = a$, or the Taylor poly nomial about $x = a$.

Notice that Taylor Polynomial of Degree n Approximating $f(x)$ for x near a will have the property that $P_n^{(m)}(a) = f^{(m)}(a)$ for $0 \leq m \leq n$.

2.2 Taylor Series

Notice that in the Taylor polynomial, if we let n here goes to infinity, we will get a series $P(x)$ with $P^{(m)}(a) = f^{(m)}(a)$ for $0 \leq m < \infty$ and thus we will expect that the series gives a good approximation about $f(x)$ around a, and actually when it converges, it is exactly the value you will get in $f(x)$, and this is called the Taylor Series.
Taylor Series for $f(x)$ about $x = 0$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = 0$, or the Taylor poly nomial about $x = 0$.

More generally, Taylor Series for $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = a$, or the Taylor poly nomial about $x = a$.

Moreover, there are several important cases that we consider, each of them is an Taylor expansion of a function about $x = 0$:

- $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots \text{ converges for all } x$$

- $$\sin(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \cdot (-1)^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ converges for all } x$$

- $$\cos(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \cdot (-1)^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \text{ converges for all } x$$

- $$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k = \sum_{k=0}^{\infty} \frac{p!}{k!(p-k)!} x^k =$$
  $$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \text{ converges for } -1 < x < 1.$$

- $$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

Moreover, we can definitely find Taylor Series based on the existing series using four methods:

- Substitude

  Example: Taylor Series about $x = 0$ for $f(x) = e^{-x^2}$

- Differentiate

  Example: Taylor Series about $x = 0$ for $f(x) = \frac{1}{(1-x)^2}$

- Integrate

  Example: Taylor Series about $x = 0$ for $f(x) = \arctan x$ (Hint: What is $\frac{d}{dx}(\arctan x)$?)

- Multiply

  Example: Taylor Series about $x = 0$ for $f(x) = x^2 \sin x$
  Example: Taylor Series about $x = 0$ for $f(x) = \sin x \cos x$
  Example: Taylor Series about $x = 0$ for $f(x) = e^{\sin x}$