MODULAR FORM

Abstract. This is a note for Jeffery Lagarias course in Modular Form.

1. Overview and History

1.1. The modular jungle.

Definition 1.1.1. Modular group is defined as $\Gamma := \Gamma(1) = PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm I$. Note that we can view Γ as a group of Mobius transformation=FLT/LFT, $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ where $\tau = x + iy \in \mathbb{H}$, acting more generally on Riemann Sphere:=$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

(Note that $\mathbb{H}$ can be viewed as hyperplane with constant negative curvature -1, and FLT act as hyperbolic isometry.)

Moreover, Γ acts on $\mathbb{H}$ discretely and the quotient have finite hyperbolic-volume $\pi/3$. (Note that the hyperbolic defect=$\pi - \alpha - \beta - \gamma$=hyperbolic area of triangle.)

Definition 1.1.2. A holomorphic modular function: $f(z)$ is a meromorphic function $f(z) : \mathbb{H} \to \mathbb{C}$ which is invariant under $PSL(2,\mathbb{Z})$, i.e. $f(\frac{a\tau+b}{c\tau+d}) = f(z)$ for every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Example 1.1.3.
1. $f(\tau) = 1$.
2. The Klein invariant $j(\tau)$.

Note that $\Gamma = PSL(2,\mathbb{Z})$ has a hyperbolic translation element $\begin{pmatrix} 1 & n \\ 1 & 0 \end{pmatrix}$, thus holomorphic modular form has period 1 and thus has an fourier expansion.

Definition 1.1.4. We also have: A (holomorphic) modular form $g(\tau)$ of weight k if it is holomorphic function such that $f(\frac{a\tau+b}{c\tau+d}) = (cz+d)^k f(z)$ (and holomorphic at the cusp.)

Example 1.1.5. Ramanujan τ-function is an arithmetic function $\tau : \mathbb{N} \to \mathbb{N}$.

$$\Delta(z) = q \prod_{i=1}^{\infty} (1 - q^n)^{24}$$

where $\Delta(z)$ is a weight $k = 12$ holomorphic cusp form. Note that holomorphic means no negative terms, and cusp form eliminate the possibility of having constant.
Now note that $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) z^n$.
Moreover, $\tau(mn) = \tau(m)\tau(n)$ (Multiplicity for $(m,n) = 1$). And $|\tau(p)| \le 2p^{11/2}$ (p is prime.)

$\Phi(s) = \int_0^{\infty} \Delta(2t) t^s \frac{dt}{t}$(Mellin Transform)=$\pi^{-s}\Gamma(s)L(s,\Delta)$, where $L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$ (Converge $\Re(s) > 12$, has Euler product and has functional equation on $\Re(s) = 12$).

1.2. Generalizaiton.

Definition 1.2.1. Γ is called a Fushion group of the first kind if Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ and the quotient $\mathbb{H}/\Gamma$ has finite hyperbolic volume.

Example 1.2.2. $\Gamma = PSL(2,\mathbb{Z})$

Definition 1.2.3.
Fundamental domain: $F = \{z \in \mathbb{H} : \text{hyper dist}(z, x) \leq \text{hyper dist}(\gamma z, x) \forall \gamma \neq id \in \Gamma\}$.

Example 1.2.4. Finite index N subgroup of $PSL(2, \mathbb{Z})$. $vol(\mathbb{H}/\Gamma) = \frac{N\pi}{3}$.

Problem: $\Gamma = < g_1, \ldots, g_k >$ finitely generated group in $PSL(2, \mathbb{R})$.
Test if Γ is a discrete group, test if it has finite volume.

1.3. **Summary of generalizations.**

1.3.1. *Vary group from $PSL(2, \mathbb{Z})$.* Consider a finite index subgroup of $SL(2, \mathbb{Z})$. Principlal congrouence subgroup $\Gamma(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N\} \subseteq SL(2, \mathbb{Z})$

Definition 1.3.1. $\Gamma \subseteq SL(2, \mathbb{Z})$ is congruence subgroup if there is some $\Gamma(N) \subseteq \Gamma \subseteq SL(2, \mathbb{Z})$, this is also called arithmetic group.

-Known: Noncongruence subgroups exists and are majority.

1.3.2. *Vary the multiplier.*

$$g(\gamma(z)) = \chi(\gamma) g(\tau) \forall \gamma \in \Gamma$$

. In particular, weight k also makes sense for some suitable k.

1.3.3. *Vary the homogeneous space.* Replace $PSL(2, \mathbb{R})$ with (reductive) arithmetic group.

Example 1.3.2. $\Gamma = Gl(n, \mathbb{Z})$ or $Gl(n, \mathbb{R})$

Example 1.3.3. Hilbert modular forms: $X = \mathbb{H} \times \mathbb{H} \times \ldots \mathbb{H}$

Example 1.3.4. Siegel modular forms: $Sp(n, \mathbb{Z})$ or $Sp(n, \mathbb{R})$

1.4. **History.** <1820 Gauss (Not published)
1831 Gauss: (Lattice version of quadratic forms)
1825-1828 Legendre: elliptic integral
1829 Abel: abelian function over abelian int (multiply periodic function on Abelian variety)
1829 Jacobi: Fundamenta Nova Theoriae Functionum Ellipticarum (Theta function.) (Inverse function of elliptic integral are elliptic function, parallel to trig function.)
1840's: Eisenstein: beautiful theory of elliptic functions (Parallel to trig).
1850's Riemann: theta function (General Riemann surface)
1854 Weierstrass: book on abelian functions.
1862 Weierstrass: Lectures in Berlin, introduce Weierstrass elliptic form.
1880's Uniformization theorem is conjectured.
Poincare: get automorphic form by averaging.
1890/1910 Trika-Klein elliptic
1890 Dedekind: sum $\Rightarrow$ Dedekind θ function.
1890 Kronecker's limit formula
1893 Hilbert: modular forms/function
1901 Otto Blumner: Hilbert modular form
1915 Ramanujan: τ function, weight 12 cusp form
1918 Hecke: Hecke operators
1939 Siegel: Modular forms attached to symplectic group (Siegel modular form)
1942 Maass: Maass forms (weak the condition to satisfies hyperbolic laplacian)

1948 Selberg Selberg trace formula
1970's Langland's program
Principle: (almost) all special functions in mathematical physics occurs as matrix coefficient of representation of "nice" Lie group over $\mathbb{R}$ or $\mathbb{C}$.
1980's connection to infinite dimensional Lie algebra.
1980's - 1990's Monster moonshine: generating function to monster simple group are modular.
2002 Mock theta functions explained by Zwegers in terms of nonholomorphic modular form.
2016 Dimension 8 sphere packing.

2. Lattice and binary quadratic forms

Start with a real 2-dimensional lattice in $\mathbb{R}^2$. Let $\Lambda = \mathbb{Z}[v, w]$ ($v = (v_1, v_2), w = (w_1, w_2)$ linear independent over $\mathbb{R}$). Here the pair of vector v and w forms an oriented basis of Lattice Λ. (Note that the basis of the lattice is not unique.) Moreover, we can denote it as B_Λ, a 2 by 2 matrix $\left(\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}\right)$.

Each basis has associated a tiling of space of $\mathbb{R}^2$ into parallelograms.

$P_{B_\lambda} := \{(x, y) = \lambda_1 v + \lambda_2 w, 0 \leq \lambda_1, \lambda_2 \leq 1\}$. Note that Λ is a discrete group inside the real Lie group $\mathbb{R}^2$, and moreover Λ is abelian, the quotient space $\mathbb{R}^2/\Lambda \simeq 2-$dim real torus. Moreover, the function on the torus lifes to the function on the cover $\mathbb{R}^2$ which is doubly periodic function.

Here, we will say P_{B_λ} is a fundamental domain for the action of group Λ (Compact quotient).

2.0.1. *Invariants of a 2-dimensional lattice.* One of the invariant of a 2-dimensional lattice is $|\det(\Lambda)| =: covol(\Lambda) =$ Area of fundamental domain $= |det\left(\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}\right)|$.

Back up: Invariant of an oriented lattice basis is $det\left(\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}\right)$= oriented area.

We can think about change of basis of lattice Λ by having a $U \in GL(2, \mathbb{Z})$ acting on the left (i.e. $B'_\Lambda = UB_\Lambda$).

2.0.2. *Another invariant: set of equal length of vectors in lattice.*

Definition 2.0.1. To a basis matrix as defined before, we can associate a quadratic form $f(x, y) = ax^2 + 2bxy + cy^2$ given by the matrix of the scalar product.

Gram matrix

$$G_\Lambda = B_\Lambda B_\Lambda^T = \left(\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}\right)^T = \left(\begin{smallmatrix} <v,v> & <v,w> \\ <w,v> & <w,w> \end{smallmatrix}\right) =: \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$$

Remark 2.0.2. By a polarization identity there is an symmetric bilinear form:

$$f(m, n) = am^2 + 2bmn + cn^2 = [mn]\left(\begin{smallmatrix} v_1 & v_2 \\ w_1 & w_2 \end{smallmatrix}\right)[mn]^T = (mv + nw)^2$$

which is the square length of lattice vector.

Notice that the polarization identity: $||x + y||^2 - ||x - y||^2 = 4 < x, y >$

Note: If one change the basis so that $B_n \mapsto UB_n$ for $U \in GL(2, \mathbb{Z})$, then the distance quadratic form changes.

$$G_{\Lambda'} \mapsto UG_\Lambda U^T$$

However, if we change the lattice by an euclidean isometry, i.e. rotate the lattice by an element $Q \in \mathcal{O}(2,\mathbb{R})$, we see that it does not changes the distance quadratic form.

$$G_\Lambda = B_\Lambda B_\Lambda^T = B_{Q\Lambda} B_{Q\Lambda}^T = G_{Q\Lambda}$$

Moreover, we denote a symmetric bilinear form associated to the quadratic form to be $M_f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Note that if M_f is associated to some lattice matrix, then $det(M_f) = ac - b^2 > 0$ since $det(M_f) = det^2(B_n) > 0$.

Moreover, for the quadratic form $f(x,y)$, we can define the discriminant of f as

$$Disc(f) = 4(b^2 - ac) < 0$$

And M_f is positive definite matrix iff $a > 0$ and $det(M_f) > 0$

2.0.3. *Correspondence between the two dimension lattice and Equivalence class of quadratic form.*

Definition 2.0.3. Call two positive definite binary form to be (Lagrange) equivalent $f_1 \sim f_2$ if there is $U \in GL(2,\mathbb{Z})$ to be the change of basis matrix such that

$$f_1(x,y) = f_2((x,y)U)$$

(Note that if they are equivalent, then the distance quadratic form of them represent exact same vaslue).

Definition 2.0.4. Call two positive definite binary form to be (Gauss) equivalent $f_1 \sim f_2$ if there is $U \in SL(2,\mathbb{Z})$ to be the change of basis matrix such that

$$f_1(x,y) = f_2((x,y)U)$$

Theorem 2.0.5. *There is a (natural) bijection*

$$\{Lattices\ in\ \mathbb{R}^2/O(2,\mathbb{R})\} \Leftrightarrow \{GL(2,\mathbb{Z}) - equivalence\ class\ of\ positive\ quadratic\ form\ M_f\}$$

Proof. Check the onto. Given positive definite quadratic form $f = ax^2 + bxy + cy^2$. Find the lattice Λ such that $M_f = B_\Lambda B_\Lambda^T$.
Note that $f(1,0) = a \Rightarrow v = (\sqrt{a}, 0)$, $f(0,1) = c \Rightarrow$ lattice vector of length c. get $<v,w>= b$. Therefore, $\cos\phi = \frac{b}{\sqrt{ac}}$.

Can do this if $|\frac{b}{\sqrt{ac}}| \leq 1 \iff b^2 \leq ac$. Uniqueness is not checked. ☐

2.1. **Reduction theory.**

Theorem 2.1.1. *Given any positive definite quadratic form. There exists a form $g = Ax^2 + 2Bxy + Cy^2$ which is $GL(2,\mathbb{Z})$ equivalent to f with $0 \leq 2B \leq A \leq C$. (Lagrange reduced form.)*

Given any positive definite quadratic form. There exists a form $g = Ax^2 + 2Bxy + Cy^2$ which is $SL(2,\mathbb{Z})$ equivalent to f with $0 \leq |2B| \leq A \leq C$. (Gauss reduced form.)

Remark: This is almost unique.

Proof. Given $f(x,y)$ find lattice Λ have $f(x,y)$ as its distance quadratic form. $\Lambda = \mathbb{Z}[v,w]$ where $||v||^2 = a$, $||w||^2 = c$, $<v,w>= b$.

Choose the basis of the lattice to $\mathbb{Z}[v',w']$ such that v' is shortest nonzero vector in lattice and w' is second shortest such that they are linearly independent.

Claim: $f' = B_{\Lambda'} B_{\Lambda'}^T$ has the required property. ☐

3. Supplements

3.1. **Uniformization of Riemann Surfaces.** A (compact) complex manifold is (compact) Riemann surface. All compact Riemann surfaces are complete nonsingular algebraic curves. (Chow's theorem: n-dimensional curve for compact n-dimensional compact manifold.)

Theorem 3.1.1. *(Uniformization theorem)*

Given a compact Riemann surface R, its universal cover is $\widehat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})(g = 0)$, $\mathbb{C}(g = 1)$, or $\mathbb{H}(g \geq 2)$.

Moreover, the covering map lifts $\pi_1(R)$ to Γ acting on the universal covering such that $R = C/\Gamma$ where C is the universal cover for R.

Consequence: Algebraic function on R lifts functions to modular function on $\mathbb{H}/\Gamma$.

Example 3.1.2. Case where $g = 1$. Under this case the uniformization gives that $E(\mathbb{C}) = \mathbb{C}/\Lambda$, where if we rescale it we get that $\Lambda = \mathbb{Z}[1, \tau]$ for some $\tau \in \mathbb{H}$.

We can also view the elliptic curve as the algebraic curve, i.e. $y^2 = x^3 + g_2(\tau)x + g_3(\tau)$. Where we see that Γ parametrize the genus 1 surface and that g_2 and g_3 are modular form for $PSL(2, \mathbb{Z})$ of weight 4 and weight 6.

3.2. **Poisson summation formula.** Poisson summation formula: sum of values of a function $f(x)$ on a lattice points $\Lambda \subseteq \mathbb{R}^n$ = sum of values of Fourier transform $\hat{f}(\xi)$ on lattice points of a dual lattice $\Lambda^* \subseteq (\mathbb{R}^n)^*$ (Multiplied by a normalization constant independent of the function $f(x)$).

Provide the function nice enough.

$$\sum_{x \in \Lambda} f(x) = C \sum_{\xi \in \Lambda^*} \hat{f}(\xi)$$

Where here C is a normalizing constant depends on the definition of dual lattice and definition of Fourier transform.

Definition 3.2.1. (Schwartz functions: nice ones)

Let $f(x_1, \ldots x_n) \in S(\mathbb{R}^n)$(Schwartz space), which are C^∞-function of n variables that rapidly decreases as $|x| \to \infty$, and all the partial derivatives rapidly decreasing.

Example 3.2.2. (Prototypical example.)

Gaussian: $f(x_1, \ldots, x_n) = e^{-x_1^2 + \ldots x_n^2}$.

Note that the quadratic terms on the top can be any general positive definite quadratic form.

Remark 3.2.3. Closure of spae of Gaussian + with translation and differentiation + linear transformation of variable = Schwartz space.

Definition 3.2.4. Fourier transform.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n}^{f(x) e^{2\pi i < x, \xi >} dx}$$

Where $< x, \xi >= x_1\xi_1 + \ldots x_n\xi_n$ real scalar product.

x = time variable (particle)

ξ = Frequency variable (wave)

Theorem 3.2.5. *The Fourier transform $f(x) \mapsto \hat{f}(\xi)$ take the Schwartz space to itself.*

Definition 3.2.6. (Lattice)

Let $\Lambda = \mathbb{Z}[v_1, \dots, v_n]$ be an n dimensional lattice in $\mathbb{R}^n$. Then

$$|\det(\Lambda)| = |\det[v_1, \dots, v_n]| = \det|V^t V|^{1/2} = vol(|\mathbb{R}^n/\Lambda|) > 0$$

The dual lattice (or reciprocal lattice $\Lambda^* \subseteq \mathbb{R}^n$)
$\Lambda^* = \mathbb{Z}[w_1^*, \dots, w_n^*]$ with $\Lambda^* = \{w^* \in \mathbb{R}^n :< v, w^* >\in \mathbb{Z} \forall v \in \Lambda\}$.

Λ^* is then spanned by the dual basis $(w_1^*, \dots, w_n^*)$, where $< w_i^*, v_j >= \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$.

Theorem 3.2.7. *Poisson summation formula, For $f(x) \in S(\mathbb{R}^n, x)$, Λ is lattice, then $\hat{\xi} \in S(\mathbb{R}^n, \xi)$. Then with the fourier transformation $e^{2\pi i <\xi, x>}$ have:*

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\det(\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y)$$

Example 3.2.8. For $\Lambda = \mathbb{Z}$ then $\Lambda^* = \mathbb{Z}$ with fourier transforma normalized $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x \xi} dx$, $\det(\Lambda) = 1$.

Proof. General case can be reduced to dim 1 and we do the work there. □

Remark 3.2.9. Consider 1-dimensional case. Can weaken hypothesis on test functions $f(x)$ allowed in Poisson summation formula to be C^2-functions, with sufficiently rapid decay at ∞.

Example 3.2.10. (Bad example) $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ integrals on both sides converge absolutely but do not agree.

Key ingredients: The poisson respects translation, multiplication by $e^{2\pi\alpha x}$, and linear transformaiton of $\mathbb{R}^n$.

Note that it is one of the basic ingredients in the proof of Jacquet langlands.

4. Elliptic function: Weierstrass $\wp$ function

Given lattice $\Lambda = \mathbb{Z}[\omega_1, \omega_2]$.

Definition 4.0.11.

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} [\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}]$$

Here we think about $\frac{1}{\omega^2}$ as the convergence factor, term must be grouped to get the uniform convergence on compact subsets of $\mathbb{C}$ (avoiding poles on the lattice Λ).

Theorem 4.0.12. *$\wp(z, \Lambda)$ is an even elliptic function, singularities are double poles on Λ.*
It is of order 2. (2 poles = 2 zeroes on the fundamental parallelogram).

Proof. Want to show uniform convergence on compact subspace of $\mathbb{C}$ on D_R for fixed R.

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}, |w| < 2R} [\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}] + \sum_{\omega \in \Lambda \setminus \{0\}, |w| > 2R} [\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}]$$

Note the first two terms are finite.
Also $|z - \omega| \geq |\omega| - |z|$. Take $|\omega| > 2R \geq 2|z|$, $z \in B_R$. In particular, we have $|\omega - z| > \frac{|\omega|}{2}$ for ω.
Have equation.

$$|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}| = |z||\frac{z\omega - z}{(z-\omega)^2\omega^2}| \leq R(\frac{5/2|w|}{|w|^4}/4) \leq \frac{10}{|\omega|^3}$$

Now boundary lemma applies we have that $\sum f_\omega(z)$ with $|f_{\omega(z)}| \leq \frac{10}{|\omega|^3}$ for $z \in B_R$, $\sum \frac{10}{|\omega|^3} < \infty$ by boundary lemma.

Thus we have the uniform convergence of sum on B_R, and thus we get a holomorphic function in B_R for $\wp(z, \Lambda, R)$.

Moreover, due to the fact that under the symmetry $z \mapsto -z$, the formula remains the same, and thus we obtained the fact that it is an even function.

Double periodicity

Claim 4.0.13.

$$\wp'(z,\Lambda) = -2(\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3})$$

differentiation term by term of the grouped factors.

Here no problem of the convergence and manifestly is doubly periodic.

This is doubly periodic by the "average" construction, i.e. $\wp(z,\Lambda) = -2(\sum_{\omega\in\Lambda} f(z-\omega))$ where $f(z) = \frac{1}{z^3}$.

Claim 4.0.14. First look at the $f_\omega(z) := \wp(z+\omega,\Lambda) - \wp(z,\Lambda)$ for given $\omega \in \Lambda$.

We then have that $f'_\omega(z) = 0$ by the double periodicity.

So $f_\omega(z) = c_\omega$ for some constant.

Now use the fact that $\wp$ is an even function, let $z = -\frac{\omega_1}{2}$, then we have that

$$c_{\omega_1} = \wp(\omega_1/2,\Lambda) - \wp(-\omega_1/2,\Lambda) = 0$$

. Similarly,

$$c_{\omega_2} = \wp(\omega_2/2,\Lambda) - \wp(-\omega_2/2,\Lambda) = 0$$

.

Thus we get the double periodicity of $\wp$. □

4.1. Laurent expansion of $\wp(z,\Lambda)$.

Theorem 4.1.1.

$$\wp(z,\Lambda) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2n+2}(\Lambda)z^{2n}$$

Where for all $m \geq 3$ we can se that

$$G_m(\Lambda) = \sum_{\omega\in\Lambda\setminus\{0\}}\frac{1}{\omega^m}$$

which converges.

Note that

(1) $G_{2n+1}(\Lambda) \equiv 0$ since $\wp$ is even function.
(2) Write $\Lambda = \omega_2\mathbb{Z}[\tau, 1]$ where $\tau = \omega_1/\omega_2 \in \mathbb{H}$.
    Then we have that

$$G_{2n}(\Lambda) = \frac{1}{(\omega_2)^{2n}}G_{2n}(\mathbb{Z}[\tau,1])$$

$G_{2n}(\tau) := G_{2n}(\mathbb{Z}[\tau, 1])$ turns out to be a weight $2k$ holomorphic modular form for $\Gamma = PSL(2, \mathbb{Z})$.

Proof. (Power series manipulation.)

Note that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n$$

converges on $|z| < 1$ uniformly on $|z| < 1 - \epsilon$.

If $\omega \neq 0$, we has that

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2}\sum_{n=0}^{\infty}(n+1)(z/\omega)^n = \sum_{n=0}^{\infty}(n+1)(z^n/\omega^{n+2})$$

So now the congruence factor appears

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \sum_{n=1}^{\infty}(n+1)\frac{z^n}{\omega^{n+2}}$$

Take $|z|$ small, near 0, then we have that $|z| < |\omega|$ for all $\omega \in \Lambda \setminus \{0\}$.

Then we have that

$$\wp(z,\omega) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\sum_{n=1}^{\infty}(n+1)\frac{z^n}{\omega^{n+2}} = \frac{1}{z^2} + \sum_{n=1}^{\infty}\sum_{\omega\in\Lambda\setminus\{0\}}(n+1)\frac{z^n}{\omega^{n+2}} = 1/z^2 + \sum_{m=1}^{\infty}(m+1)G_{2m+2}(\Lambda)z^{2n}$$

using Tonelli-Fubini, and the odd term vanishes since it is even function. □

Theorem 4.1.2. *The function $\wp(z,\Lambda)$ satisfies the nonlinear first order differential equation, let $y = \wp(z,\Lambda)$*

$$(y')^2 = 4y^3 - g_2(\Lambda)y - g_3(\Lambda)$$

, where $g_2(\Lambda) = 60G_4(\lambda)$, $g_3(\Lambda) = 140G_6(\lambda)$

Proof. Treat Λ fixed.

Elliptic function $f(z)$ are meromorphic in $\mathbb{C}$ and satisfies that $\frac{\partial}{\partial \bar{z}}f(z) = 0$.

Operator $\frac{\partial}{\partial z}$ preserves the property of the elliptic function.

Since $f(z+\omega) = f(z)$, thus we have that $f'(z+\omega) = f'(z)$.

Now look at the function

$$f(z) := (y')^2 - (4y^3 - g_2(\Lambda)y - g_3(\Lambda))$$

which is an even function.

These function has pole in the fundamental domains only at $z = 0$, where $(y')^2$ has a 6th order pole , and y^3 has a 6th order pole, both at $z = 0$. So we can take conbination of 1 , $(y')^2$, y^3, y to kill terms z^{2k} where $k = 0, 1, 2, 3$ in the Laurent expansion at $z = 0$, and thus we get a bounded elliptic function with $f(0) = 0$, thus $f \equiv 0$.

Now we have that $A\wp'(z)^2 + B\wp(z)^3 + C\wp(z) + D \equiv 0$. Now we write down exactly what the Laurent expansion, we will have that $(\wp'(z))^2 - 4\wp^3(z) = -\frac{60G_4}{\wp}(z) - 140G_6 + ...$

Thus done. □

Satisfy second order differential equation $\wp''(z) = 6(\wp)^2 - \frac{1}{2}g_2(\Lambda)$.

Corollary 4.1.3. *Set* $\omega_3 = \omega_1 + \omega_2$*, Let* Λ *be the fixed lattice.*

Let $e_i = \wp(\omega_i/2, \Lambda)$*,* $i = 1, 2, 3$*.*

Then 1) $4x^3 - g_2(\Lambda)x - g_3(\Lambda) = 4\prod_{i=1}^{3}(x - e_i)$*. All three roots are distinct.*

2) Let $z_0 \neq \Lambda$*, then we have* $f(z) = \wp(z, \Lambda) - \wp(z_0, \Lambda)$ *has exactly* 2 *distinct zeroes* $z = \pm z_0($ mod $\Lambda)$*, unless* z_0 *is a* 2*-dim point where it has a double zero at* $z = z_0$*.*

Proof. Note that $\wp'(z)$ is odd function.

$$\wp'(e_i, \Lambda) = -\wp'(e_i, \Lambda)$$

Thus $\wp'(e_i) = 0$. These are 3 distinct zeros of $\wp'(z)$, thus all the zeros of $\wp'(z)$ are simple. □

4.2. 2-division points.

Corollary 4.2.1. *Let* $\wp(w, \Lambda)$ *for* $\Lambda = \mathbb{Z}[\omega_1, \omega_2]$*,*

(1) *Set* $\omega_3 := \omega_1 + \omega_2$ *on period parallelogram.*
Set

$$e_i = \wp(\omega_i/2, \Lambda)$$

for $i - 1, 2, 3$*, these are the* $2-$*division points, then* $4X^3 - g_2(\Lambda)X - g_3 = 4\prod(x - e_i)$ *and all three roots are distinct.*

$$g_2(\Lambda) = G_4(\Lambda)$$

$$g_3(\Lambda) = 140G_6(\Lambda)$$

where $G_{2k}(\Lambda) = \sum' \frac{1}{\omega^{2k}}$

(2) *Let* $z_0 \notin \Lambda$*, then the function* $f(z) = \wp(z, \Lambda) - \wp(z_0, \Lambda)$ *has exactly two distinct zeros, and* $z \equiv \pm z_0 \mod \Lambda$*, unless* z_0 *is a* 2*-division point* (z_1, z_2, z_3) *when it has a double zero at* $z = z_0$*.*

Proof. (1) Have differential equation

$$(\wp(z)')^2 = 4\wp(z)^3 - g_2(\Lambda)y - g_3(\Lambda)$$

So want to show that $\wp'(w\omega_i/2) = 0$, by evenness we have that $e_i = \wp(\omega_i/2)$ is a zero of the equation $4X^3 - g_2X - g_3$, and e_i are distinct. Since first that ω_i, $(i = 1, 2, 3)$ are the only distinct zeros of $\wp'(z) \mod \Lambda$, then we consider $f_i(z) = \wp(z) - e_i$ is an elliptic function of order 2, and it is even since $\omega_i/2$ is 2-division point, then $f_i(z)$ has double zero at $\omega_i/2$ and no other zeros, and thus we have that e_i distinct.

(2) Suppose $z_0 \notin \Lambda$, and $z_0 \in \frac{1}{2}\Lambda$, then we have that $\wp(z, \Lambda) - \wp(z_0, \Lambda) \equiv \wp(z, \Lambda) - \wp(-z_0, \Lambda)$ is even elliptic function of order 2.

If clearly has a zero at $z = z_0$ and $z = -z_0$, $z_0 \not\equiv -z_0 \mod \Lambda$, so have 2 zeros and all of them since it is order 2. If $z_0 \in \frac{1}{2}\Lambda$, then has a double zero at $z = z_0$, accounts for all zeros. □

Proposition 4.2.2. *For a fixed lattice* Λ*, any elliptic function for* Λ *is a rational function in* $\mathbb{C}(x, y)$ *pf* $x = \wp(z)$*,* $y = \wp'(z)$*, subject to relation* $y^2 = 4x^3 - g_2x - g_3$*.*

The field K *of elliptic function is isomorphic to the field of fraction of* $\mathbb{C}[x, y]/(y^2 - (4x^3 - g_2x - g_3))$

Proof. Reduction to even function case. Since $f(z)$ is elliptic function for Λ, then so is $f(-z)$ (since lattice $\Lambda = -\Lambda$). So is $g(z) = \frac{1}{2}(f(z) + f(-z)$ even function, and $h(z) = \frac{1}{2}(f(z) - f(-z)$.

Since $\wp(z)$ is even function and $\wp'(z)$ is odd function, then $\frac{h(z)}{\wp'(z)}$ is even function.

So it suffice to construct $\frac{h(z)}{\wp'(z)}$ and $g(z)$ both are even function, $\Rightarrow$ cab recover $f(z)$ from these two function.

We have reduced problem to construct all even elliptic function.

Claim: All even elliptic unction $g(z)$ is rational function of $\wp(z)$, i.e. $g(z) = R(\wp(z))$.

Proof of the claim: build up an elliptic function $\tilde{h}(z)$ having the same zeros and poles as $h(z)$, all of the lattic Λ where $\tilde{h}(z)$ is built using translation of $\wp(z) - \wp(z_0)$ for various z_0.

Suppose we not constructed $\tilde{h}(z)$, then we see that $h(z)/\tilde{h}(z)$ is an elliptic function with all zeros and poles on the lattice Λ, must be of order zero since it has no poles nor zeros, thus its nonzero constant k, then $h(z) = k(\tilde{h}(z))$, done

Recipe: the order of zero or pole of $h(z)$ at $z = z_0$ denote as $\nu_{z_0}(h)$.Fudge factor at $z_0 \in \Lambda$, $w(z_0) = 2$ if $2z_0 \notin \Lambda$, $w = 1$ if $2z_0 \in \Lambda$.

Let $\tilde{h(z)} = R(\wp(z)) = \frac{N(\wp(z))}{D(\wp(z))}$, goal: kill zeros and poles Z at a time, which we can do since $h(z)$ is an even function.

Take

$$N(z) = \prod_{z\text{: zeroes of } h, \text{ in period parallelagram.}} (\wp(z) - \wp(z_0))^{\frac{\nu_{z_0}(h)}{w(z_0)}})$$

$$D(z) = \prod_{z\text{: poles of } h, \text{ in period parallelagram.}} (\wp(z) - \wp(z_0))^{\frac{-\nu_{z_0}(h)}{w(z_0)}})$$

Observe that $N(z)$ and $D(z)$ have poles only on the lattice Λ.

Now $\tilde{h}(z)$ is an even elliptic form has the certain property that when $w(z_0) = 2$, then z_0, $-z_0$ are both zeros, and $2z_0 \notin \Lambda$.

Factors combine in pair to

$$(\wp(z) - \wp(z_0))^{\frac{\nu_{z_0}(h)}{w(z_0)}})(\wp(z) - \wp(-z_0))^{\frac{-\nu_{z_0}(h)}{w(z_0)}})$$

has a zero of multiplicity 1 at both z_0 and $-z_0$, other wise for the 2-division point $(\wp(z) - \wp(z_0))^{\frac{\nu_{z_0}(h)}{1}})$ works. □

4.3. **Fourier expansion of $G_{2k}(\Lambda)$ and Eisenstein Series.**

Definition 4.3.1. For even $2k \geq 4$, we have holomorphic Eisenstein series

$$G_{2k}(\Lambda) = \sum{}'(1/\omega^{2k}) = \frac{1}{\omega_2^{2k}} G_{2k}(\mathbb{Z}[\tau, 1])$$

, note that this $1/\omega_2^{2k}$ is a constant depends on choice of the basis. Note that holomorphic means holomorphic function of τ on $\mathbb{H}$.

Set $G_{2k}(\mathbb{Z}[\tau, 1]) =: 2G_{2k}(\tau) = 2\sum' \frac{1}{c\tau + D}^{2k}$, Called holomorphic Eisenstein series.

It is invariant under $\tau \mapsto \tau + 1$ and preserves the lattice Λ when $z = x + iy$ then $G_{2k}(\tau)$ has a Fourier expansion in the x variable, with y as the parameter.

Definition 4.3.2. For $\tau \in \mathbb{H}$, $2k \geq 4$ even integer. The weight $2k \in 2\mathbb{Z}$ nonholomorphic Eisenstein series.

$$G_{2k}(s,\tau) = \frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2}' \frac{y^s}{(c\tau+d)^{2k}|c\tau+d|^s}$$

And the weight $2k$ holomorphic Eisenstein series:

$$\frac{1}{2}G_{2k}(\mathbb{Z}[\tau,1]) = G_{2k}(\tau) := \frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2}' \frac{1}{(c\tau+d)^{2k}}$$

where we have $\tau = x + iy$ so $y > 0$. that the series converges absolutely and uniformly to an complex analytic function of s for fixed $\tau \in \mathbb{H}$, $\Re(s) > 1 - K$.

This function is real analytic in τ(non-holomorphic): $y = \Im(\tau)$ is not a holomorphic on $\mathbb{H}$.

It satisfies a Laplacian type pde, Eigen function of a cer tain hyperbolic Laplacian in variable x and y.(elliptic pde).

$\Delta_k = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - 2ky(\frac{\partial^2}{\partial x})$.

Fact:

(1) $\Delta_K(G(\tau,s)) = s(s-1)E(\tau,s)$ eigenfunction of Laplacian Δ_K eigenvalue continuously varies with s.
(2) $G(\tau,s)$ satisfies a functional equation. Fix τ takes $s \mapsto c_k - s$ after multiplying by some $\Gamma(s)$ factor.

If you specialize to point $s = 0$, we then recover the holomorphic Eisenstein seriesm since we have that $G_{2k}(0)(\tau) = G_{2k}(\tau)$

For $s = 0$, we have that $\Delta_K G_{2k}(\tau) = 0$ "holomorphic condition".

Note that Eisenstein series also makes sense for $k = 2, 0, -2, \ldots$. For $\Re(s)$ big enough, $\Re(s) > 1 - k$, however, the point $s = 0$ is not in the correspondence domain for $2k \leq 2$.

Note: Selberg did prove the analytic continuation and functional equation. One way to do it is to compute the Fourier expansion of nonholomorphic Eisenstein series in x-variable. It transforms nicely under $PSL(2,\mathbb{Z})$.

Claim: invariant under $\tau \mapsto \tau + 1$ in the lattic $\mathbb{Z}[\tau,1]$, $G_{2k}(s,\tau) = G_{2k}(s,\tau+1)$, where $\Re(s) > 1 - k$.

Proposition 4.3.3. *Fourier expansion of* $G_{2k}(\tau)$.

$$G_{2k}(\tau) = \zeta(2k) + \frac{(2\pi \mathrm{i})^{2k}}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi \mathrm{i} n\tau}$$

Definition 4.3.4. Normalized $G_{2k}(\tau)$,

$$E_{2k}(\tau) := \frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2,\gcd(c,d)=1}' \frac{1}{(c\tau+d)}^{2k}$$

Reason: Formula look nicer unfolding.

$$G_{2k}(\tau) = \sum_{m-1}^{\infty}\frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2,\gcd(c,d)=m}' \frac{1}{(c\tau+d)}^{2k} = \sum_{m=1}^{\infty}\frac{1}{2}\frac{1}{m^{2k}}\Big(\sum_{(c,d)\in\mathbb{Z}^2,\gcd(c,d)=1}' \frac{1}{(c\tau+d)}^{2k}\Big).$$

Note that $\sum'_{(c,d)\in\mathbb{Z}^2,\gcd(c,d)=1}\frac{1}{(c\tau+d)}^{2k} = E_{2k}(\tau)\sum_{m=1}^{\infty}\frac{1}{m^{2k}} = \zeta(2k)E_{2k}(\tau)$

Fourier expansion of $E_{2k}(\tau)$,

$$E_{2k}(\tau) = 1 + \frac{(-1)^k}{(2k-1)!}\frac{(2\pi)^{2k}}{\zeta(2k)}\sum_{k=1}^{\infty}\sigma_{2k-1}(k)e^{2\pi \mathrm{i}\tau}$$

Note that $e^{2\pi \mathrm{i}n\tau} = e^{2\pi inx}e^{-2\pi ny}$

Proposition 4.3.5. *$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}}(\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n)$ where $q := e^{2\pi \mathrm{i}\tau}$, positive parameter. And $0 \le |q| < 1$ makes sense.*

Definition 4.3.6. Due to the above, we have that

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)e^{-2piny}e^{2\pi inx}$$

where $e^{2\pi iy}$ is the Whittaker function.

Good news: it is holomorphic in q

Bad news: not modular, thus quasimodular

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi y}$$

Good news: modular of weight 2

Bad news: not holomorphic in τ.

Quasimodular property:

$$E_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2E_2(\tau) + \frac{12c(c\tau+d)}{2\pi \mathrm{i}}$$

This is the solution to the Ramanujan mock theta function.

Certain q-series that kind of behave like theta functions with modular form property, holomorphic q-expansion.

Solution due to Zagier: add a non-holomorphic piece ("shadow") and resulting modular form (non holomorphic).

4.4. **"Fourier Series" expansion of $\wp(z,\Lambda)$.** Suppose $\Lambda = \mathbb{C}[\omega_1,\omega_2]$ lattice.

Let $\tau = \omega_1/\omega_2$, with $Im(\tau) > 0$.

$q = e^{2\pi \mathrm{i}\tau}$ ($\tau \in \mathbb{H}$ means $0 < |q| < 1$, use invariance under $\tau \to \tau+1$ to make q-variable well defined.)

$u = e^{2\pi \mathrm{i}z/\omega_2}(u \in \mathbb{C})$.

Double periodic: invariance under $z \mapsto \omega_1 + z$, $z \mapsto \omega_2 + z$ gives $q \mapsto q$, $u \mapsto qu$, and $q \mapsto q$, $u \mapsto u$. Fourier expansion in powers of q.

Theorem 4.4.1. *("Fourier" expansion of $\wp$ and $\wp'$)*

(1)

$$\wp(z,\Lambda) = (\frac{2\pi \mathrm{i}}{\omega_2})^2\{(\frac{1}{12} + \frac{u}{(1-u)^2}) + \sum_{n=1}^{\infty}q^n[\frac{u}{(1-q^nu)^2} + \frac{1}{(q^n-u)^2} - \frac{2}{(1-q^n)^2}]\}$$

("Pade approximation")

(2)

$$\wp'(z,\Lambda) = (\frac{2\pi \mathrm{i}}{\omega_2})^3\{\frac{1+u}{(1-u)^3} + \sum_{n=1}^{\infty}q^n[\frac{1+q^nu^3}{1-q^nu} + \frac{q^n+u}{(q^n-u)^3}]\}$$

Remark 4.4.2. (1) Formula for $\wp$ gives that for $\wp'$ by applying $\frac{d}{dz}$.

(2) Double periodicity almost visible.
$z \mapsto z + w_2$ holomorphic since q, u does not change
$z \mapsto z + w_1$ says changes u to qu and hope the expansion becomes equal to itself.(Exercise)

(3) This is a horrible opaque formula.

(4) $z = 0$ gives that $u = 1$ and thus we can see that $\frac{u}{(1-u)^2}$ is the double pole at $z = 0$.

Proof. Introduce and study a magic function which corresponds to $\wp(z)$.

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2n+2}(\Lambda)z^{2n}$$

Study magic function

$$Z(z, \Lambda) := \sum_{n=1}^{\infty} G_{2n}(\Lambda_\tau)z^{2n}$$

Note that :

$$(\frac{Z(z) - G_2^*(\Lambda)z^2 - 1}{z})' = \wp(z)$$

Step 1:
Claim:

$$\frac{Z(z)-1}{z} = \frac{2\pi \mathrm{i}}{\omega^2}(\frac{1+u}{2(1-u)} + \sum_{n=1}^{\infty} q^n(\frac{u}{1-q^n u} - \frac{1}{u-q^n}) = \frac{2\pi \mathrm{i}}{\omega^2}(\frac{1+u}{2(1-u)} + \sum_{n=1}^{\infty} q^n(\frac{1}{1/u - q^n} - \frac{1}{u-q^n})$$

Step 2:
Assume the claim and derive the Fourier series.
Note that we have $du/dz = \frac{2\pi \mathrm{i}}{\omega_2}u$ and $dq/dz = 0$.

$$\wp(z) + G_2^*(\Lambda) = (\frac{2\pi \mathrm{i}}{\omega_2})^2(-\frac{1}{2(1-u)} + \frac{(1+u)}{2(1-u)^2} + \sum_{n=1}^{\infty} q^n(\frac{u}{(1-q^n u)^2} + \frac{1}{1-q^n u} + \frac{1}{(u-q^n)^2})$$

Now we shift G_2^* to the right side and we use the fourier expansion of G_2^*

Lemma 4.4.3.

$$G_2^* = (\frac{2\pi \mathrm{i}}{\omega_2})^2(-\frac{1}{12} + \sum_{m=1}^{\infty} \sigma_1(m)q^{mn}) = (\frac{2\pi \mathrm{i}}{\omega_2})^2(-\frac{1}{12} + \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}) = (\frac{2\pi \mathrm{i}}{\omega_2})^2(-\frac{1}{12} + \frac{q^n}{(1-q^n)^2})$$

Proof of the lemma:
Note that $\sigma_1(m) = \sum_{d|m} d$ = sum of divisor of $m = \sum_{de=m} d = \sum_{de=m} e$.
Now we can get result by interchanging d and e.

$$\sum_{m=1}^{\infty} \sigma_1(m)q^m = \sum_{m=1}^{\infty}\sum_{m=de} dq^{de} = \sum_{e=1}^{\infty}(\sum_{d=1}^{\infty} dq^{de}) = \sum_{e=1}^{\infty} \frac{q^e}{(1-q^e)^2} = \sum_{d=1}^{\infty} \frac{dq^d}{1-q^d} = \sum_{d=1}^{\infty}(\sum_{e=1}^{\infty} dq^{de})$$

and then we plug in $\zeta(2) = \pi^2/3$, we are done. □

And using the lemma into our result we have the desired thing.
Step 3: Prove the claim.

$$\begin{aligned} Z(z) &= \sum_{n=1}^{\infty} G_{2n}(\Lambda) z^{2n} \\ &= \sum_{n=1}^{\infty} \frac{z^{2n}}{\omega_2^{2n}} (2\zeta(2n) + 1\frac{(2\pi \mathrm{i})^{2n}}{(2n-1)!} \sum_{m=1}^{\infty} \sigma_{2n-1}(m) q^m) \\ &= 2\sum_{n=1}^{\infty} \zeta(2n)(\frac{z}{\omega_2})^{2n} + 2\sum_{m=1}^{\infty} q^m (\sum_{n=1}^{\infty} \sigma_{2n-1}(m)(\frac{2\pi \mathrm{i} z}{\omega_2})^{2n} \frac{1}{(2n-1)!}) \end{aligned}$$

Proposition 4.4.4.

$$\sum_{n=1}^{\infty} \zeta(2n) x^n = \frac{1 - \pi x \cot(\pi x)}{2} = \frac{1 - \mathrm{i}\pi x (\frac{e^{2\pi \mathrm{i} x}+1}{e^{2\pi \mathrm{i} x}-1})}{2}$$

Proof of the above identity is left as hw, related to the partial fraction expansion of cotangent which is

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty} (\frac{1}{z+m} - \frac{1}{z-m})$$

Suppose this is true hen we can just take $x = \frac{z}{\omega_2}$,

$$2(\sum_{n=1}^{\infty} \zeta(2n)(\frac{z}{\omega_2})^{2n}) = 1 - (\frac{\mathrm{i}\pi z}{\frac{u+1}{u-1}})$$

Now we are done with the first term.

For the second term. For fix m, q^m term

$$\begin{aligned} \sum_{n=1}^{\infty} \sigma_{2n-1}(m)(\frac{2\pi \mathrm{i} z}{\omega_2})^{2n} \frac{1}{(2n-1)!} &= \sum_{n=1}^{\infty} \sum_{d|m} d^{2n-1} (\frac{2\pi \mathrm{i} z}{\omega_2})^{2n} \frac{1}{(2n-1)!} \\ &= \sum_{d|m} \sum_{n=1}^{\infty} d^{2n-1} (\frac{2\pi \mathrm{i} z}{\omega_2})^{2n} \frac{1}{(2n-1)!} \\ &= \frac{2\pi \mathrm{i} z}{\omega_2} \sum_{d|m} \sum_{n=1}^{\infty} d^{2n-1} (\frac{2\pi \mathrm{i} d z}{\omega_2})^{2n-1} \frac{1}{(2n-1)!} \\ &= \frac{2\pi \mathrm{i} z}{\omega_2} (\sum_{d|m} \sinh(\frac{2\pi \mathrm{i} z}{\omega_2})) \\ &= \frac{2\pi \mathrm{i} z}{\omega_2} \sum_{d|m} (\frac{u^d - u^{-d}}{2}) \end{aligned}$$

Now we interchange the d and e summation, we have the second term to be

$$\begin{aligned} &\frac{2\pi \mathrm{i} z}{\omega_2} \sum_{d|m} (\frac{u^d - u^{-d}}{2})(\sum_{e=1}^{\infty}) \\ &= \frac{2\pi \mathrm{i} z}{\omega_2} \sum_{d=1}^{\infty} \sum_{e=1}^{\infty} (q^e u)^d - q^{de} u^{-d} \\ &= \frac{2\pi \mathrm{i} z}{\omega_2} \sum_{e=1}^{\infty} q^e (\frac{u}{1 - q^n u} - \frac{1}{u - q^e}) \end{aligned}$$

And now we are done. □

5. Quasi elliptic function

5.1. Weierstrass ζ-function: Quasi elliptic function.

Definition 5.1.1. Weight 2 holomorphic eisenstein series before.

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

$$G_2(\omega_1,\omega_2) = \frac{\pi^2}{3}\frac{1}{\omega_2^2}E_2(\tau)$$

Theorem 5.1.2. *(1)There is a unique meromorphic function $\zeta(z,\Lambda)$ satisfying $\zeta'(z,\Lambda) = -\wp(z,\Lambda)$ which is an odd function (at $z = 0$).*

(2)It is meromorphic, simple poles on lattices and

$$\zeta(z,\Lambda) = \frac{1}{z} + \sum_{\omega\in\Lambda}{}'\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right) = \frac{1}{z^2} + z^2\left(\sum_{\omega\in\Lambda}\frac{1}{\omega^2(z-\omega)}\right)$$

(3)Quasiperiodic

$$\zeta(z + m_1\omega_1 + m_2\omega_2) = \zeta(z,\omega) + m_1\eta_1 + m_2\eta_2$$

where $\eta_1,\eta_2 \in \mathbb{C}$ are quasi periods.

In fact, $\eta_i = 2\zeta(\omega_i/2,\Lambda)$ for $i = 1,2$.

(4)Fourier expansion

Note that we still use the notation that $q = e^{2\pi i\tau}$, $u = e^{\frac{2\pi iz}{\omega_2}}$

$$\zeta(z,\Lambda) = \frac{\pi^2}{3\omega_2^2}E_2(\tau)z - \frac{2\pi i}{\omega_2}\left(\frac{1+u}{2(1-u)} + \sum_{n=1}^{\infty}q^n\left(\frac{u}{1-q^nu} + \frac{1}{q^n-u}\right)\right)$$

where $\frac{\pi^2}{3\omega_2^2}E_2(\tau)z$ is the connecting factor for $E_2(\tau)$ and $G_2(\omega_1,\omega_2)$

(5) Formula for quasiperiod as a function of τ.

$$\eta_1 = \frac{\pi^2}{3\omega_2}\tau E_2(\tau) - \frac{2\pi i}{\omega_2}$$

$$\eta_2 = \frac{\pi^2}{3\omega_2}E_2(\tau)$$

and satisfies the Legendre-Weierstrass relation:

$$\omega_1\eta 2 - \omega_2\eta_1 = 2\pi i$$

Proof. (1) and (2):
Define $H(z)$ =RHS of the equation. Check the converges uniformly on the compact subsets. Differentiate term by term formula in (2), we have $\frac{\partial}{\partial z}H(z) = \wp(z)$. Check it is an odd function.

(3):
Give ω_Λ. Look at $f_\omega(z) = f(z+\omega) - f(z)$ for $f(z) = \zeta(z,\omega)$. Also we have $f_\omega'(z) = f'(z+\omega) - f'(z) + \wp(z+\omega) + \wp(z) \equiv 0$.

Thus we have that $f\omega(z) = constant = \eta_\omega$. Define

$$\eta_1 = \zeta(z+\omega_1) - \zeta(z)$$

$$\eta_2 = \zeta(z+\omega_2) - \zeta(z)$$

Set $z = -\omega_i/2$ and get the basis case, then we induct on m, n, we then get the desired result.

(4)Fourier expansion.

Claim: $\frac{Z(z)-1-G_2^*(\Lambda)z^2-1}{z} = \zeta(z,\Lambda)$, and we are then done.

(5)Fourier expansion to get value for η_1, η_2.

Start with η_2, use variable u in the fourier expansion is invariant under $z \mapsto z+\omega_2$, then we have

$$\begin{aligned}\eta_2 :\equiv\, &\zeta(z+\omega_2,\Lambda) - \zeta(z,\Lambda)\\ &= \frac{\pi^2}{3\omega_2^2}E_2(\tau)[(z+\omega_2)-z]\\ &= \frac{\pi^2}{3\omega_2^2}E_2(\tau)\omega_2\\ &= \frac{\pi^2}{3\omega_2}E_2(\tau)\end{aligned}$$

Legendre-Weierstrass equation: Integrate $\zeta(z,\Lambda)$ around the period parallelogram (Along $a+\omega_2$, $a+\omega_1+\omega_2$, $a+\omega_1$ counterclockwisely) and we have that

$$\frac{1}{2\pi \mathrm{i}}\oint \zeta(z)dz = \sum Residue(\zeta(z)) = 1$$

Integrate around the side we have that

$$I = 2\pi\mathrm{i} = \int_{II} + \int_{IV} + \int_{I} + \int_{III} = \omega_1\eta 2 - \omega_2\eta_1$$

Now we can solve η_1 using the Legendre-Weierstrass relation. □

5.2. **Quasimodularity of $E_2(\tau)$.**

Proposition 5.2.1. $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) + \frac{12c(c\tau+d)}{2\pi\mathrm{i}}$$

Proof. Change basis of $\Lambda = \mathbb{Z}[\omega_1,\omega_2]$, let $(\omega_1',\omega_2')^T = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)(\omega_1,\omega_2)^T$.

Key point: $E_2(\tau)$ depends only on lattice Λ through parameter τ, not an basis, but $G_2(\omega_1,\omega_2)$ does depends on the basis.

In new basis $\zeta(z+\omega_2',\Lambda)$.

$$\zeta(z+\omega_2',\Lambda) - \zeta(z,\Lambda) = \eta_2' = \frac{\pi^2}{3\omega_2'}E_2(\tau')$$

but $\omega_2' = c\omega_1 + d\omega_2$, so evaluate this we have in the old basis,

$$\zeta(z+\omega_2',\Lambda) - \zeta(z,\Lambda) = c\eta_1 + d\eta_2$$

Thus we have $c\eta_1 + d\eta_2 = \eta_2'$.

Thus we have that

$$\frac{\pi^2}{\omega_2}(c\tau+d)E_2(\tau) - c\frac{2\pi\mathrm{i}}{\omega_2} = \frac{\pi^2}{\omega_2'}E_2\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\pi^2}{\omega_2(c\tau+d)}$$

Use identity

$$\omega_2' = c\omega_1 + d\omega_2 = \omega_2(c\tau + d)$$

Here $\tau' = \frac{a\tau+b}{c\tau+d} = \frac{\omega_1'}{\omega_2'}$ in new basis.

$$(c\tau+d)^2 E_2(\tau) - c\frac{2\pi i}{\omega_2}(3\pi^2(c\tau+d)) = (c\tau+d)^2 E_2(\tau) + \frac{12c(c\tau+d)}{2\pi \mathrm{i}}$$

□

5.3. **Quasi-elliptic function: Weierstrass σ function.** Weierstrass theory of theta function.

Theorem 5.3.1. *Let $\Lambda = \mathbb{Z}[\omega_1, \omega_2]$.*

(1) *There is unique entire function $\sigma(z,\Lambda)$, the Weierstrass σ function with the following property.*

$$\frac{d}{dz}(\log(\sigma(z))) = \frac{\sigma'}{\sigma}(z;\Lambda) = \zeta(z,\Lambda)$$

$$\lim_{z\to 0}\frac{\sigma(z,\Lambda)}{z} = 1$$

$\sigma(z,\Lambda)$ has a simple zero at $z=0$.

(2) *Function $\sigma(z,\Lambda)$ is an odd function, simple zeros at all points of Λ, no other zero.*

$$\sigma(z,\Lambda) = z\prod_{\omega\in\Lambda}^{\omega\neq 0}(1-\frac{z}{\omega})e^{\frac{z}{\omega}-\frac{z^2}{2\omega^2}}$$

It is an entire function of order 2. (Infinite product that converges uniformly on compact subsets of $\mathbb{C}$)

(3) *Multiplicative quasiperiodisity.*
For any $\omega = m_1\omega_1 + m_2\omega_2 \in \Lambda = \mathbb{Z}[\omega_1,\omega_2]$

$$\sigma(z+\omega,\Lambda) = \pm e^{\eta_\omega(z+\omega/2)}\sigma(z,\Lambda)$$

Where η_ω is the quasi period of the Weiersrass ζ function.
Where sign is 1 if $\omega\in 2\Lambda$, -1 otherwise.

(4) *Fourier expansion of σ: Product expansion.*

$$\sigma(z,\Lambda) = \frac{\omega_2}{2\pi\mathrm{i}}e^{\eta_{\omega_2}/\omega_2}(u^{1/2}-u^{-1/2})\times\prod_{n=1}^{\infty}\frac{(1-q^n u)(1-q^n/u)}{(1-q^n)^2}$$

with the notation $q = e^{2\pi\mathrm{i}\tau}$, $u = e^{2\pi\mathrm{i}z/\omega_2}$ as usual.

Proof. (1), (2), (3) as exercise.

(4) Use the Fourier Expansion of $\zeta(z,\Lambda)$[Partial factor expansion]

$$\zeta(z,\Lambda) = \frac{\pi^2}{3\omega_2^2}E_2(\tau)z - \frac{2\pi\mathrm{i}}{\omega_2}(\frac{1+u}{2(1-u)} + \sum_{n=1}^{\infty}q^n(\frac{u}{1-q^n u} + \frac{1}{q^n-u})$$

We should guess that

$$\sigma(z,\tau) = Ce^{\eta_2(\frac{z^2}{\omega_2})z}(u^{1/2}-u^{-1/2})\prod(1-q^n u)$$

And then we take a log of it and take a derivative., we can determine the normalizing factor C.

$\lim_{z\to 0} \frac{\sigma(z,\Lambda)}{z} = 1$, and $u \to 1$, and $z \to 0$. Then we have that

$$C(\frac{2\pi i}{\omega_2}\prod(1-q^n)^2) = 1$$

□

Proposition 5.3.2. *Assume $a \notin \Lambda$.*

Then we get

$$\wp(z,\Lambda) - \wp(a,\Lambda) = -\frac{\sigma(z-a,\Lambda)\sigma(z+a,\Lambda)}{\sigma(a,\Lambda)^2\sigma(z,\Lambda)^2}$$

Note $\frac{d^2}{dz^2}(\log(\sigma(z,\Lambda))) = -\wp(z,\Lambda)$.

Proof. To prove the identity:

(a)Check right hand side is an elliptic function.

(b) Check poles on both sides match.

(c)Check the Laurant expansions of both sides at $z = 0$ match down h constant term.

For (a), RHS

$$\sigma(z-a+\omega_1,\Lambda) = (-1)e^{\eta_1(z-a+\omega_1/2)}\sigma(z-a,\Lambda)$$

$$\sigma(z+a+\omega_1,\Lambda) = (-1)e^{\eta_1(z+a+\omega_1/2)}\sigma(z+a,\Lambda)$$

$$\sigma^{-2}(z+\omega_1,\Lambda) = (-1)^2 e^{2\eta_1 z+\omega_1/2}\sigma^{-2}(z,\Lambda)$$

The checking for ω_2 is similar.

Therefore it is elliptic function.

For (b) RHS has double pole at $z = 0(z \in \Lambda)$. has two zeros at $z = \pm a_1$. $\sigma(z,\Lambda)$ has a zero at $z = 0$ (Only singularity), has no pole. Therefore, $\frac{1}{\sigma^2(z,\Lambda)}$ has double pole at $z = 0$.

$\sigma(z-a,\Lambda)$, $\sigma(z+a,\Lambda)$ has 2 simple zeros at $z = \pm a$, double zero at $z = \pm a$ if $2a \in \Lambda$.

Therefore, zeros and poles match.

For (c)

$$LHS = \frac{1}{z^2} + O(1)$$

around z=o.

$$RHS = (-\frac{\sigma(-a,\Lambda)\sigma(a,\Lambda)}{(\sigma(0,\Lambda))^2})\frac{1}{z^2} + O(1)$$

Since $\sigma(z)$ is an odd function, and $\lim_{z\to 0}\frac{\sigma(z,\Lambda)}{z} = 1$.

Claim: There is no $\frac{1}{z}$ term on the RHS.

Then we are done. □

6. Theta functions

Definition 6.0.3. A holomorphic function $f(z) : \mathbb{C} \to \mathbb{C}$ is a theta function for a lattice Λ if for $\lambda \in \Lambda$ it transforms multiplicatively as

$$f(z+\lambda) = e_\lambda(z)f(z)$$

Where $e_\lambda(z)$ is an entire function with no zeros.

Example 6.0.4. Reimann theta function is defined as the following:

$$\Theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi \mathrm{i}(n^2\tau+2nz)}$$

for given lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. Moreover, we have that

$$\Theta(z+1,\tau) = \Theta(z)$$

$$\Theta(z+m\tau;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi \mathrm{i}(n+m)^2\tau+2(n+m)(z)-m^2-2mz} = e^{-\pi \mathrm{i}(m^2\tau+2mz)}\Theta(z,\tau)$$

Big picture:

1. (1) Theta function expansion converge rapidly if $y = \Im(\tau)$ is large suitable for numerical computation (under modular change, we can make y large).
2. (2) Can recover elliptic function as quotients.
3. (3) Satisfies important PDE's.
4. (4) Use to embed abelian varieties in projective space.
5. (5) Get modular form at point $z = 0$. ("Theta nulls")

6.1. **Classifying theta factors.**

Lemma 6.1.1. *The theta factors satisfies the cocycle condition($Z^1(\Lambda, \mathcal{O}^*(\mathbb{C}))$):*

$$e_{\gamma+\gamma'}(z) = e_\gamma(z+\gamma')e_{\gamma'}(z).$$

Proof. Calculate two ways. ☐

Consider rescaling by

$$f(z) \mapsto \phi(z)f(z) =: g(z)$$

where $\phi(z)$ is an entire function nowhere zero. Now $\phi(z) = \exp(h(z))$.

Conclude that $g(z)$ is a theta function $g(z+\lambda) = \widetilde{e_\lambda(z)}g(z)$, with the theta factor

$$\tilde{e}_\lambda(z) = e_\lambda(z)\phi(z+\lambda)(\phi(z))^{-1}$$

So we call the change of θ factor by $\phi(z+\lambda)\phi(z)^{-1}$ a coboundary condition.

Definition 6.1.2. Call two theta function $f(z)$ and $g(z)$ equivalent if you can take on to another one by operations.

$$f(z) \mapsto f(z+\mu)$$

by translation.

or by

$$f(z) \mapsto h(z) = f(z)\phi(z)$$

where $\phi(z)$ is entire with no zeros, and we choose $\widetilde{e_\lambda(z)} = e_\lambda(z)[\phi(z+\lambda)\phi^{-1}(z)]$

Note: don't require $\phi(z)$ to have any symmetries with respect to Λ.

Proposition 6.1.3. *Any theta function on $\mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}[\tau,1]$ is equivalent to one have theta factors that are exponential of linear functions, i.e.* $\tilde{e_\lambda}(z) = e^{-2\pi \mathrm{i}(a_\lambda z+b_\lambda)}$

Proof. prove or accept as given. ☐

Lemma 6.1.4. *Any theta function on* $\mathbb{C}/\Lambda$ *is equivalent to* $e_\lambda(z) = e^{-2\pi i(a_\lambda z + b_\lambda)}$ *such that*

$$\tilde{f}(z+1) = \tilde{f}(z)$$
$$\tilde{f}(z+\tau) = e^{-2\pi i(kz+\tilde{b})}\tilde{f}(z)$$

where $k \in \mathbb{Z}$.

Proof. Step 1.

$$f(z+1) = e^{-2\pi i(z_1 z + b_1)} f(z)$$

Guess to choose

$$\tilde{f}(z) = f(z) e^{2\pi i(\frac{1}{2}a_1 z^2 - ()b_1 - \frac{1}{4})z}$$

$$\phi_1(z) := 2\pi i(\frac{1}{2}a_1 z^2 - ()b_1 - \frac{1}{4})z$$

Check by calculation $f(z+\tau) = e^{2\pi i(a_\tau z + b_\tau)}$. Now

$$f \tilde{+} 1(z) = \tilde{f}(z)$$
$$\tilde{f}(z+\tau) = e^{-2\pi i(kz+\tilde{b})}\tilde{f}(z)$$

$\tilde{e_1}(z) = \phi(z+1)\phi(z)^{-1}e_1(z) = e^{2\pi i(\frac{1}{2}a_1(z+1)^2 - (b_1 - \frac{1}{4}a_1)(z+1)} = \dots$

Second to show $\tilde{a} \in \mathbb{Z}$. □

Definition 6.1.5. For a theta function of $\Lambda = \mathbb{Z}[tau, 1]$, we call k the order of a theta factor of the form

$$f(z+1) = f(z)$$
$$f(z+\tau) = e^{-2\pi i kz - 2\pi i\tilde{b}} f(z)$$

6.2. **Construction of theta function for** $\Lambda = \mathbb{Z}[\tau, 1]$**.**

Definition 6.2.1. We define that for $k \geq 1$ and $0 \leq s \leq k-1$, we have the Riemann theta function is defined as

$$\Theta_s(z,\tau)_k := \sum_{n\in\mathbb{Z}} e^{\pi i((s/k+n)^2 k\tau)} e^{2\pi i z((s/k+m)k)} = \sum_{n\in\mathbb{Z}} (q^k)^{\frac{n+s/k}{2}} (u^k)^{n+s/k}$$

With the Riemann Theta factor $e_1(z) = 1$ and $e_\tau(z) = e^{-2\pi i kz}$

Definition 6.2.2. The vector space of order k theta function is

$$Th(k,\Lambda) := < \Theta_s, (z,\tau)_k >$$

Theorem 6.2.3. *Let* $\Lambda = \mathbb{Z}[\tau, 1]$, $\tau \in \mathbb{H}$.

(1) *Each theta function is equivalent to one having a theta factor of form*

$$e_{m+n\tau}(z) = e^{-2\pi i k(nz + \frac{n^2}{2}\tau)}$$

where $k \in \mathbb{Z}$. *Call the space of such theta function* $Th(k,\Lambda)$.

(2) *The space* $Th(k,\Lambda)$ *has dimension* $\begin{cases} 0 & k < 0 \\ 1 & k = 0 \\ k & k \geq 1 \end{cases}$

For $k \geq 1$, $Th(k,\Lambda)$ *is spanned by* $\Theta_s, (z,\tau)_k$.

Proof. Use the fourier series expansion $f(z+1) = f(z)$.

Note $e_m(z) = 1$ for $m \in \mathbb{Z}$.

It has a fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi \mathrm{i} n z}$$

Because it is a C^1- function on horizontal line $\Im(z) = y$.

$$f(x+iy) = \sum_{n=-\infty}^{\infty} c_n(y) e^{2\pi \mathrm{i} n x}$$

In fact $c_n(y) = c_n e^{-2\pi n y}$.

Anyhow, for fixed y this is a L^1-function of x, periodic with period 1, and hence $|c_n| \to 0 \equiv$ Riemann Lebesgue lemma.

Know $|c_n| = |\int_0^1 F(x) e^{-2\pi \mathrm{i} n x} dx| \le \int_0^1 |F(x)| dx \le c_0$.

But we may compute $f(z+\tau) = e^{-2\pi \mathrm{i}(k\tau + b)} f(z)$. Compute the fourier series in two ways.

$f(z+\tau) = \sum_{n=-\infty}^{\infty} (c_n e^{2\pi n \tau} e^{2\pi \mathrm{i} n z})$

Another way, we can have that $f(z+\tau) = \sum_{n=-\infty}^{\infty} (c_n e^{-2\pi n \tilde{b}} e^{2\pi \mathrm{i}(n-k)z}))$

Since k is integerm then we can shift the indices and have that

$$c_n e^{2\pi \mathrm{i} n \tau} = c_{n+k} e^{-2\pi \mathrm{i} \tilde{b}}$$

and this gives a recurrence fir c_n.

Now we consider the case $k = 0$, we have that

$$c_n e^{2\pi \mathrm{i} n \tau} = c_n e^{-2\pi \mathrm{i} \tilde{b}}$$

and thus since $\tilde{b}$ is fixed and τ is also fixed, so at most one n can work.

Therefore, we have $f(z,\tau) = c e^{2\pi \mathrm{i} N z}$, now we just renormalized this so that $f(\tilde{Z}) = f(z) e^{-2\pi \mathrm{i} n z}$, and now we have $e_{m+n\tau=1}$ for all m, n.

Case $k < 0$, suppose some $c_n \neq 0$, then if we suppose also $n \geq 0$, then

$$c_{n+k} e^{-2\pi \mathrm{i} \tilde{b}} e^{2\pi \mathrm{i} n \tau} = c_n \neq 0$$

We repeat the term j times, we will have that $c_{n+jk} e^{-2\pi \mathrm{i} \tilde{b}} e^{j(j-1)/2 * 2\pi \mathrm{i} k \tau} = c_n$ with $\tau = u + iv$, for $j > 0$, as $j \to \infty$, we have that the polynomial is really large.

Note that we can run the similar argument for the other side.

In the case where $k > 0$, we will get

$$c_{n+jk} e^{-2\pi \mathrm{i} \tilde{b}} e^{-j(j-1)/2 * 2\pi \mathrm{i} k \tau} = c_n$$

and $e^{-j(j-1)/2 * 2\pi \mathrm{i} k \tau}$ is very large since $\Re(-2\pi \mathrm{i}(\frac{j(j-1)}{2} k\tau)) = +2\pi(\frac{j(j-1)}{2} k \Im(k) > 0$

Get k linearly independent terms since the periodicity depends on the modulo class mod k. Now we get set of fourier coefficient is arithmetic progression, and there are k such solutions, and note that thus they are linearly independent, we have the vector space of dimension k.

Furthermore, we can rescale the theta function by translation, therefore, we can rescale $f(z) \mapsto f(\tilde{z}) = f(z + \tau/2 - \tilde{b}/k)$, and we have that $f(\widetilde{z+1}) = f(\tilde{z})$, $\tilde{f}(z+\tau) = e^{-2\pi \mathrm{i}(kz + \frac{k\tau}{2})} \tilde{f}(z)$ □

Alternative: The following note is the proof by Dolgachev on the same theorem.

Proof. There are two kinds of proof available.
One of them is the AG proof, based on the definition and the Riemann Roch theorem, we have that $\dim H^0(L) = k$ since $\deg L = k$ on an elliptic curve.
Another is the barehand proof.

Since $f(z+1) = f(z)$, thus we have an Fourier expansion for $f(z) = \sum c_n q^n$.
Compare the Fourier expansion we have for $f(z)$ and $f(z+\tau)$, we have that $c_{n+k} = c_n e^{\pi i(2n-k)\tau}$, and moreover $c_{j+sk} = e^{\pi i(j+sk)^2\tau/k}$.

Therefore

$$f(z) = \sum_{j=0}^{k-1} c_j \theta_j(z,\tau)_k,$$

where $\Theta_j(z,\tau)_k := \theta_{j/k,0}(kz, k\tau)$ □

Remark 6.2.4. Note that k is the number of the zeros and the zeros lies on the points $\frac{1}{2}\tau + \frac{1}{2k} + \frac{s}{k}\tau + \frac{i}{k}$ for $i = 0, \ldots, k-1$.

Lemma 6.2.5. *There is a bilinear pairing of Riemann theta function*

$$Th(k,\Lambda) \times Th(k',\Lambda) \to Th(k+k',\Lambda)$$

Proof. Check action on theta factors. □

6.3. **Theta function with characteristics(Jacobi Theta funciton.)** The effect of translation in z-variable on a theta function gives a new theta function (With same number of zeros), but its theta factor changes.

Definition 6.3.1. The theta function with rational characteristics $a, b \in \mathbb{Q}$ is

$$\theta_{ab}(z,\tau) := \sum_{s\in\mathbb{Z}} e^{\pi i((a+s)^2\tau + 2(z+b)(s+a)}$$

The lattice is $\mathbb{Z}[1,\tau]$ and the theta factors depends on (a,b).

Note that:

- a, $b \in \mathbb{C}$ should still makes sense.
- Riemann theta function $\Theta_s(z,\tau)_k = \theta_{s/k,0}(kz, k\tau)$

Definition 6.3.2. We can then define the four basic Jacobi theta function as the following:

$$\theta_1(z,\tau) := \theta_{1/2,1/2}$$
$$\theta_2(z,\tau) := \theta_{1/2,0}$$
$$\theta_3(z,\tau) := \theta_{0,1/2}$$
$$\theta_0(z,\tau) := \theta_{0,0}$$

And θ_1 is odd function while the others are even.

Moreover,

$$Z(\theta_{1/2,1/2}(z,\tau)) = 0 + \Lambda = \Lambda$$
$$Z(\theta_{0,0}(z,\tau)) = 1/2 + \tau/2 + \Lambda = \Lambda$$
$$Z(\theta_{0,1/2}(z,\tau)) = \tau/2 + \Lambda = \Lambda$$
$$Z(\theta_{1/2,0}(z,\tau)) = 1/2 + \Lambda = \Lambda$$

Lemma 6.3.3. (1)
$$\theta_{a,b}(z,\tau) = e^{2\pi i a(b-b')}\theta_{a',b'}(z,\tau) \textit{ for } a'-a, b'-b \in \mathbb{Z}$$
(2)
$$\theta_{a,b}(z,\tau) = e^{2\pi i a}\theta_{a',b'}(z,\tau)$$
(3)
$$\theta_{a,b}(z,\tau) = e^{-2\pi i b}e^{-\tau-2z}\theta_{a',b'}(z,\tau)$$
(4)
$$\theta_{a,b}(z,\tau) = e^{\pi i(a^2\tau+2(z+b)a)}\Theta(z+b+a\tau)$$

Lemma 6.3.4. *There is a bilinear pairing of spaces of theta function $Th(k,\Lambda_\tau)_{a,b}$*
$$Th(k,\Lambda)_{ab} \times Th(k',\Lambda)_{a'b'} \to Th(k+k',\Lambda)_{(a+a')(b+b')}$$
Where $Th(k,\Lambda)_{ab} := \{f(\tau) : e_{m+n\tau = e^{-2\pi i(nb-ma)}}\}$
And we have the transition formula applies on the space nicely as well.

6.4. **Hesse cubic embedding.** There is an embedding $E = \mathbb{C}/\Lambda \to (\Theta_0, \dots, \Theta_{k-1})(k \geq 3)$. In the case where $k=3$, we have that it lies on the Hesse cubic: $x^3+y^3+z^3+\gamma xyz = 0$.

This descent from the lifted map from $\mathbb{C} \to \mathbb{C}^3$.

Observation, there is a group action on $\mathbb{C}^3$ of order 27, which is the Heisenberg group of order 27.

$G = \{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\}$

It has a center which is $Z(G = (I, \zeta_3 I, \zeta_3^2 I))$, and the projective group $G/Z(G) \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ of order 9, and it acts on the Hesse cubic on $\mathbb{P}^2(\mathbb{C})$.

6.5. **Theta null=theta function.** Set $z=0$, for Riemann theta function $\Theta(0,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}u^n = 1 + 2(\sum_{n=1}^{\infty} q^{n^2})$

Theorem 6.5.1. *(Theta product formula)*
$$\Theta_0(z,\tau) = Q(q)\prod_{m=1}^{\infty}(1+q^{\frac{2m-1}{2}}e^{2\pi i z})(1+q^{\frac{2m-1}{2}}e^{-2\pi i z})$$
for an analytic function $Q(q)$ defined in $|q|<1$ with $Q(0)=1$.

Proof. Take $P(z,q) = \prod_{m=1}^{\infty}(1+q^{\frac{2m-1}{2}}u)(1+q^{\frac{2m-1}{2}}u^{-1})$.

We want to show the zeros are $\frac{1+\tau}{2}+\Lambda$. Note that the zeros are at $2(z^*) = (1+2m)\tau+(1+2n)$.

Now we exponentiate we get ,
$$e^{\pi i 2z^*} = (-1)e^{\pi i\tau(2m-1)}$$

Now, we see that if $m \geq 1$, we recover all zeros of first factor, if $m \leq 0$, we recover the zeros in the second factor.

Therefore, we can conclude that $\frac{\Theta_0(z,\tau)}{P(z,q)}$ is a function that has no zeros, and we want it is a constant $Q(q)$.

Periodic with period 1 in z variable, thus claim that we can show that it is bounded in the strip and then we can compute by Liouville theorem.

□

Remark 6.5.2. Our goal is to determine $Q(q)$. Actually, we know that $Q(q) = \prod_{n=1}^{\infty}(1-q^n)$, this is almost a modular form.

The modular form is $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ which is a 1/2 modular form with a multiplier system $(\frac{12}{n})$ which is Kronecker symbol and is periodic of modulo 24.

6.6. **Jacobi triple product formula.** $\Lambda = \mathbb{Z}[\tau, 1]$ as usual, $q = e^{\pi i\tau}$, $u = e^{2\pi i z}$

Theorem 6.6.1. *For $|q| < 1$, $u \in \mathbb{C} \smallsetminus \{0\}$, there is a formula*

$$\prod_{n=1}^{\infty}(1-q^n u)\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-q^n/u) = \sum_{k=0}^{\infty}(-1)^k(u^k - u^{-(k+1)})q^{\frac{k(k+1)}{2}}$$

And further more we have the right side

$$\sum_{k=0}^{\infty}(-1)^k(u^k - u^{-(k+1)})q^{\frac{k(k+1)}{2}} = \sum_{-\infty}^{\infty}(\frac{-4}{n})q^{\frac{n^2-1}{2}}u^{\frac{n-1}{2}}$$

And moreover,

$$\prod_{n=1}^{\infty}(1+q^{\frac{n-1}{2}}u)(1+q^{\frac{n-1}{2}}/u)(1+q^n) = \sum_{n=-\infty}^{\infty}q^{n^2/2}u^n$$

as in the last theorem.

Corollary 6.6.2. *Weierstrass σ function for $\Lambda = \mathbb{Z}[\omega_1, \omega_2]$, and $\tau = \frac{\omega_1}{\omega_2}$.*

$$\sigma(z, \Lambda) = \frac{\omega_2}{2\pi i}e^{\eta_2 z^2/(2\omega_2)}(\frac{\sum_{k=}}{den})$$

(Missing, need Angus.)

Proof. Plug into the Fourier expansion. □

Aside,

Theorem 6.6.3. *(Finite Jacobi triple product identity.)*

$$(1-u)\prod_{n=1}^{N}(1-q^n u)\infty(1-q^n/u) = \sum_{k=-N}^{N}(-1)^k(u^k)q^{\frac{k(k-1)}{2}}\prod_{n=1}^{k+N}(\frac{1-q^{2(N+1)-n}}{1-q^n})$$

6.7. **Jacobi's Theorem+Corollary.**

Definition 6.7.1. A theta null, or theta constant is $\theta(0, \tau)$ where $\theta(z, \tau)$ is the theta function.

Now we view it as a function of τ. We will show $\Theta_{00}(0, \tau)$ behaves like a weight 1/2 modular form.

Now we define $\theta(\tau) := \theta_{00}(0, \tau)$.

We then have

$$\theta(\frac{a\tau + b}{c\tau + d}) = A(cz + d)^{1/2}\theta(z)$$

for any matrix in $\Gamma_0(4)$.

Now we see that all the four basic theta function except $\theta_{1/2,1/2}(0, \tau)$ is modular form with weight 1/2, and $\theta'_{1/2,1/2}(0, \tau)$ is also a modular form of weight 3/2.

Theorem 6.7.2.

$$(-\pi)\theta_{00}(0,\tau)\theta_{0,1/2}(0,\tau)\theta_{1/2,0}(0,\tau) = \theta'_{1/2,1/2}(0,\tau)$$

Remark 6.7.3. In the proof show that theta null equation satisfies

$$\frac{\theta'''_{1/2,1/2}(0,\tau)}{\theta'_{1/2,1/2}(0,\tau)} - \frac{\theta'''_{0,1/2}(0,\tau)}{\theta'_{0,1/2}(0,\tau)} = \frac{\theta'''_{1/2,0}(0,\tau)}{\theta'_{1/2,0}(0,\tau)} + \frac{\theta'''_{0,0}(0,\tau)}{\theta'_{0,0}(0,\tau)}$$

Theorem 6.7.4. *Other Jacobi theta function have the product that*

$$\theta_{0,1/2}(z,\tau) = Q(q)\prod_{m=1}^{\infty}(1 - q^{\frac{2m-1}{2}}e^{2\pi \mathrm{i}z})(1 + q^{\frac{2m-1}{2}}e^{-2\pi \mathrm{i}z})$$

$$\theta_{1/2,0}(z,\tau) = Q(q)q^{1/8}(e^{\pi \mathrm{i}z} + e^{-\pi \mathrm{i}z})\prod_{m=1}^{\infty}(1 + q^m e^{2\pi \mathrm{i}\tau})(1 + q^m e^{-2\pi \mathrm{i}\tau})$$

$$\theta_{1/2,1/2}(z,\tau) = Q(q)q^{1/8}(e^{\pi \mathrm{i}z} - e^{-\pi \mathrm{i}z})\prod_{m=1}^{\infty}(1 - q^m e^{2\pi \mathrm{i}\tau})(1 - q^m e^{-2\pi \mathrm{i}\tau})$$

Corollary 6.7.5. *We then have the following formula:*

$$\theta_{0,0}(0,\tau) = Q(q)\prod_{m=1}^{\infty}(1 + q^{\frac{2m-1}{2}})^2$$

$$\theta_{0,1/2}(0,\tau) = Q(q)\prod_{m=1}^{\infty}(1 - q^{\frac{2m-1}{2}})^2$$

$$\theta_{1/2,0}(0,\tau) = Q(q)2q^{1/8}\prod_{m=1}^{\infty}(1 + q^m)^2$$

$$\theta_{1/2,1/2}(0,\tau) \equiv 0$$

$$\theta'_{1/2,1/2}(0,\tau) = -Q(q)2\pi q^{1/8}\prod_{m=1}^{\infty}(1 - q^m)^2$$

6.8. **Modularity of Theta null products.** Let $\theta(\tau) = \theta_{0,0}(0,\tau)$, Let $\theta_k(\tau) = \theta_{0,0}(0,\tau)^k$.

Theorem 6.8.1.

$$\theta(\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)\tau) := \theta_k(\tau + 2) = \theta_k(\tau)$$

$$\theta_k(\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\tau) := \theta_k(-\frac{1}{\tau}) = (-\mathrm{i}\tau)^{k/2}\theta_k(\tau)$$

This states that θ_k is weight $k/2$ modular forms on $\Gamma(2) =< -I, \left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right)>$

Proof. It suffices to prove for $k = 1$ since θ_k functional equation follows by exponentiation.

Note that (1) is clear.

(2) is proved using the Poisson summation formula applied to lattice $\Lambda = \mathbb{Z}$, $\dim(\Lambda) = 1$, $\tau = \mathrm{i}y$ where $y > 0$, $f_y(x) = e^{\pi \mathrm{i}x^2(iy)} = e^{-\pi x^2 y}$.

Consider the test function $f(x) = e^{\pi \mathrm{i}x^2\tau} = e^{-\pi x^2 y} \in S(\mathbb{R})$.

Use the poisson, $\Lambda = \mathbb{Z}$, $\Lambda^* = \mathbb{Z}$, $\det(\Lambda) = 1$.

$\theta_{00}(0,iy) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = \sum_{n\in\mathbb{Z}} f_y(n) = \sum_{m\in\mathbb{Z}} \hat{f}_y(m) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e^{2\pi \mathrm{i}xm}dx = \sum_{m\in\mathbb{Z}} \int e^{-\pi x^2 y}e^{2\pi \mathrm{i}xm}dx = \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{y}}\int_{-\infty}^{\infty} e^{-\pi(\tilde{x} - \frac{im}{\sqrt{y}})^2}d\tilde{x}e^{\pi m^2/y} = \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{y}}e^{-\pi m^2/y} = \frac{1}{\sqrt{y}}\theta_{00}(0,\frac{\mathrm{i}}{y})$, thus done with a small gap. ∎

Note that we now see that $\theta(0,\tau$ transform as a weight 1/2 modular form on $\Gamma(2)$ with multiplier system. (Actually it lives a conguence group, the theta group Γ_θ)

Jacobi theta function are entire function in z variable where the modular parameter $q = e^{2\pi i\tau}$ has $|q| < 1$, holomorphic in both (z,q) in this domain.

They have nice infinite product expansions and we know the zeros. Assuming lattice contains 1, we get infinite products in q variables.

Also Jacobi theta function has rapidly convergent expansion in the q variables. In general for fixed z, $f(q,\tau) = \sum a_n q^n$ convergent for $q < 1$, but jacobi theta has Lacuna expansion which means a large block of a_n to be 0.

Jacobi theta functions satisfies important diff eq, Heat equation/KdV-equation.

7. Euler identity

Theorem 7.0.2. *Euler's identity.*

$$\prod_{m=1}^{\infty}(1-q^m) = \sum_{r\in\mathbb{Z}}(-1)^r q^{r(3r+1)/2}$$

Remark 7.0.3. The first part is related to the theta function and the right is related to some triangular numbers.

The right side encodes a kind of fourier expansion of Dedekind η function, taking $q = e^{2\pi i\tau}$

$$\eta(\tau) := q^{1/24}\prod_{m=1}^{\infty}(1-q^m) = q^{1/24}\sum_{r\in\mathbb{Z}}(-1)^r q^{r(3r+1)/2}$$

where $r(3r+1)/2$ is the pentagonal number.

Proof. Expand $\theta_{1/6,1/2}(0,3\tau)$ two ways.

1.

$$\theta_{1/6,1/2}(0,3\tau) = \sum_{m\in\mathbb{Z}} e^{\pi i(m+1/6)^2(3\tau)+0+2(m+1/6)1/2} = e^{\pi i/6}e^{\pi i\tau/2}\sum_{m\in\mathbb{Z}}(-1)^m q^{(m^2+3m)/2}$$

2.

$$\theta_{1/6,1/2}(0,3\tau) = e^{\pi i/6}e^{\pi i\tau/2}\theta_{0,0}(1/2+\tau/2,3\tau) = e^{\pi i/6}e^{\pi i\tau/2}\prod_{m=1}^{\infty}(1-e^{6\pi i m\tau})\prod_{m=1}^{\infty}(1-e^{(6m+1)\pi i\tau})(1-e^{(6m+2)\pi i m\tau})$$

□

8. Importance of Dedekind eta function.

-Theta function forms a family where τ varies. They form a real analytic family in τ parameter, we have seen product formulas for jacobi theta which require a scaling factor $Q(q)$ outside the infinite product which is essentially Dedekind eta.

-Show up in the transformation laws for theta function under modular transformation.

-Show up in real analytic modular forms, the real analytic eisenstein series of weight zero.

$$E(\tau,s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{y^s}{|m\tau+n|^{2s}}$$

Here $E(\tau,s)$ is real analytic in $\tau\in\mathbb{H}$, complex analytic in s variable and is converge for $\Re(s) > 1$.

Moreover,

$$\Delta E(z,s) = s(s-1)E(z,s)$$

where $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, the hyperbolic laplacian.

Fact:

1. For $\tau \in \mathbb{H}$, $E(\tau, s)$ analytically continuous to $s \in \mathbb{C}$ to a meromorphic function, with simple poles at $s = 1, 0$, residues $-\pi/3$
2. Functional equation. $\hat{E}(\tau, s) = \hat{E}(\tau, 1 - s)$, where $\hat{E}(\tau, s) := \pi^{-s}\Gamma E(\tau, s)$
3. Modular form with weight zero for $PSL(2, \mathbb{Z})$

4, Moderate growth at the cusp

- Fourier series expansion at cusp.

$$E(\tau, s) = CT() + \sum_{n \neq 0} a_n(s) W_{k,s-1/2}(4\pi(n)y)e^{2\pi nx}$$

Where $W_{k,\mu}$ is the Whittaker function (confluent hypergeometric function.).
And CT satisfies the funcitonal equation and almost RH.

- $\eta(\tau) = \mathfrak{q}^{1/24} \prod_{n=1}^{\infty}(1 - q^n) = q^{1/24}Q(q)$.

Proposition 8.0.4. *(Kronecker's first limit formula)*
Laurent expansion at $s = 1$ of $E(\tau, s)$

$$E(\tau, s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log(2) - \log(\sqrt{y}|\eta(\tau)|^2))$$

8.1. **Dedekind eta function.**

Definition 8.1.1. $\eta(\tau) = \mathfrak{q}^{1/24} \prod_{n=1}^{\infty}(1 - q^n) = q^{1/24}Q(q)$.

Theorem 8.1.2. *Dedekind eta function is a weight $\frac{1}{2}$ modular form on $\Gamma(1)$ with multiplier system:*

(1)

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau)$$

(2)

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$$

Remark 8.1.3. Dedekind actually show that $\eta(\frac{a\tau+b}{c\tau+d}) = \epsilon(a, b, c, d)(c\tau + d)^{1/2}\eta(\tau)$.

Where, we have $\epsilon(a, b, c, d) = e^{2\pi\frac{a+d}{12c} - s(d,c) - 1/4}$ for $c = 1$, where $s(d, c)$ is the Dedekind sum, i.e. $s(h, k) := \sum_{n=1}^{k-1} \frac{n}{k} B_1(\frac{hn}{k})$.

Proposition 8.1.4. *Jacobi theta nulls expression in terms of eta function.*

$$\theta_{0,0}(\tau) = \frac{\eta(\tau)^5}{\eta(\tau/2)^2\eta(2\tau)^2}$$

$$\theta_{0,1/2}(\tau) = e^{\frac{2\pi i}{24}} \frac{\eta(\tau)^5}{\eta((\tau + 1)/2)^2\eta(2\tau)^2}$$

$$\theta_{1/2,0}(\tau) = 2\frac{\eta^2(2\tau)}{\eta(\tau)} = 2e^{\frac{2\pi i}{24}} \frac{\eta(\tau)^5}{\eta((\tau + 1)/2)^2\eta(2\tau)^2}$$

$$\theta_{0,0}(\tau) = -2\pi(\eta(\tau)^3)$$

8.2. **Transformation laws for theta function $Th(k,\Gamma_\tau)_{a,b}$ with characters.** Given Γ_τ as before. Now definition of theta function with characteristics applies for a fixed τ. For τ, $Th(k,\Gamma_\tau)_{a,b}$ has a nice basis of function, which naturally extend to all $\tau \in \mathbb{H}$, and we get such a continuous extension on τ for all functions in $Th(k,\Lambda_\tau)_{a,b}$.

Theorem 8.2.1. *Let $f(z,\tau) \in Th(k,\Gamma_\tau)_{a,b}$ be level k thetafunction with characteristics. Consider $\tau' = M\tau = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$. Then*

$$g(z,\tau) = e^{\pi i\gamma(\gamma\tau+\delta)z^2} f(z(\gamma\tau+\delta),\tau) \in Th(k,\Gamma_{\tau'})_{a',b'}$$

where the new characteristics are

$$a'' = \delta a - \gamma b + k\frac{\gamma\delta}{2} \mod 1$$

$$b'' = -\beta a + \alpha b - k\frac{\alpha\beta}{2} \mod 1$$

Dolgachev has a different but equivalent version in his note, which is not written down here.

Proof. Claim 1:(Dilations)

If $f(z,\tau) \in Th(e_\tau,\Gamma_\tau)$, then for any $t \in \mathbb{C}^*$, $\phi(z,\tau) = f(\frac{z}{t},\tau) \in Th(e_{\lambda'}, t\Lambda_\tau)$ where $e'_\lambda(z) = e_{\frac{\lambda'}{t}}(z/t)$

$$\phi(z+t\tau, t\Lambda_\tau) = f(\frac{z+t\lambda}{t},\Lambda_\tau) = f(z/t+\lambda,\Lambda_\tau) = e_\lambda(z/t)\phi(z,t\Lambda_\tau) = e_{\lambda'/t}(z/t)\phi(\tau,t\Lambda_\tau)$$

.

Claim 2:

For M and τ' as in the question, and $t = \frac{1}{\gamma\tau+\delta}$. Then have $t\Lambda_\tau = \Lambda_{\tau'}$

Direct calculation shows it.

Claim 3:

Apply t and $\lambda' \in \Lambda'_\tau$.

Claim 1 gives $f(z(\gamma\tau+\delta),\tau) \in Th(e_{\lambda'}, t\Lambda_\tau) = Th(e_{\lambda'(\gamma\tau+\delta)}, \Lambda_{\tau'})$.

Compute for $f \in Th(e,\Lambda_\tau)_{a,b}$.

... Check photo for note.

□

Application:

Theorem 8.2.2. *Action $\tau \to \tau+1$.*

(1)
$$\theta_{1/2,1/2}(z,\tau+1) = -e^{\pi i/4}\theta_{1/2,1/2}(z,\tau)$$

(2)
$$\theta_{0,0}(z,\tau+1) = \theta_{0,1/2}(z,\tau)$$

(3)
$$\theta_{0,1/2}(z,\tau+1) = \theta_{0,0}(z,\tau)$$

(4)
$$\theta_{1/2,0}(z,\tau+1) = e^{-\pi i/4}\theta_{1/2,0}(z,\tau)$$

Theorem 8.2.3.

(1)
$$e^{-i\pi z^2/\tau}\theta_{1/2,1/2}(z/\tau,-1/\tau)=(\sqrt{i\tau})^3\theta_{1/2,1/2}(z,\tau)$$
(2)
$$e^{-i\pi z^2/\tau}\theta_{0,0}(z/\tau,-1/\tau)=\sqrt{i\tau}\theta_{0,1/2}(z,\tau)$$
(3)
$$e^{-i\pi z^2/\tau}\theta_{0,1/2}(z/\tau,-1/\tau)=\sqrt{i\tau}\theta_{0,0}(z,\tau)$$
(4)
$$e^{-i\pi z^2/\tau}\theta_{1/2,0}(z/\tau,-1/\tau)=\sqrt{i\tau}\theta_{1/2,0}(z,\tau)$$

Proof. Step 1: Transformation law
Step 2: Match the correct a' and b'
Step 3: Set $z=0$ and use information of theta nulls. □

Then we can derive a corollary.

Corollary 8.2.4. *Let $f(\tau)=\theta'_{1/2,1/2}(0,\tau)$, then for any matrix M in $SL(2,\mathbb{Z})$*
$$f(\frac{\alpha\tau+\beta}{\gamma\tau+\delta})=\phi(M)(\gamma\tau+\delta)f(\tau)$$
where $\phi(M)^8=1$ is some 8-th root of unity.

In q-series, $\theta'_{1/2,1/2}(0,\tau)=-2\pi\eta(\tau)^3$

Proof. Suffice to check for the map $\tau\mapsto\tau+1$ and $\tau\mapsto-1/\tau$, apply the first two theorem and do derivative and we are done. □

Remark 8.2.5. Theta transformation law works best for the matrix with $\alpha\beta\equiv 2 \mod 0$, $\gamma\delta\equiv 0 \mod 2$, this forms a subgroup of $\Gamma(1)$, which is $\Gamma_\theta=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \mod 2\}$

Corollary 8.2.6. *Assume $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\theta$ theta group. Then we have that*
$$\theta_{0,0}(z(\gamma\tau+\delta)^{-1},(\alpha\tau+\beta)(\gamma\tau+\delta)^{-1})\phi(\tilde{M})(\gamma\tau+\delta)^{1/2}e^{\pi i\gamma z^2/(\gamma\tau+\delta)}$$

Proof. Similar to the last corollary. □

9. Modular forms form Jacobi form

Theorem 9.0.7. *(DiDimensionvision point Laurent series)*

Let $\Phi(z,\tau)$ be a meromorphic function of z, (doubly) periodic in z for the lattice $\Lambda_\tau=\mathbb{Z}[1,\tau]$ of finite index.

$\Phi(\frac{z}{\gamma\tau+\delta},\frac{\alpha\tau+\beta}{\gamma\tau+\delta})=(\gamma\tau+\delta)^m\Phi(z,\tau)$ *for* $\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\in\Gamma\subset\Gamma(1)$ *(Transform like a modular form of weight m, but can have poles.)*

Now at $z_0=x\tau+y$, let $\Phi(z,\tau)$ has a Laurent expansion $\Phi(z,\tau)=\sum_{n=-k}^{\infty}g_n(\tau)(z-z_0)^m$

Then we get modularity of g_n for some $M\in SL(\mathbb{Z})$ such that
$$(*)(x',y')=(x,y)M\equiv(x,y)\mod\mathbb{Z}^2$$

Here $(*)$ holds for some congruence subgroup of Γ with $(x,y)\in\mathbb{Q}^2$.

Proof. By Cauchy theorem.$g_n(\frac{\alpha\tau+\beta}{\gamma\tau+\delta})=\frac{(\gamma\tau+\delta)^m}{2\pi i}2\pi i(\gamma\tau+\delta)^n g(\tau)$. Details see Dolgachev's note. □

9.1. **Weak modularity.**

Definition 9.1.1. A function $f(\tau)$ is call weakly modular of weight k on a group $\Gamma \subset PSL(2,\mathbb{Z})$.

If $(f|_k M)(\tau) = f(\tau)$ for all $M \in \Gamma$ where $(f|_k M) := (\gamma\tau+\delta)^{-k} f(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}) = j(M,\tau)^{k/2} f(\frac{\alpha\tau+\beta}{\gamma\tau+\delta})$.
It is modular if it is meromorphic at cusp and holomorphic in $\mathbb{H}$.

Proposition 9.1.2. *The theta constants $\theta^4_{0,0}(\tau)$, $\theta^4_{1/2,0}(\tau)$, $\theta^4_{0,1/2}(\tau)$, $\theta^4_{1/2,1/2}(\tau)$ are weak modular form of weight 2 on $\Gamma(2)$*

Proof. Can extract this from the propositions proved before which are the theta identities. then calculate using the modular transform of them over generators of $\Gamma(2)$. □

Three examples:
Γ, $\Gamma_t heta$, $\Gamma(2)$ as before. Consider: (holomorphic) modular form (like differential form), modular function (weight 0).

Remark 9.1.3. -Generally infinite dual vector space on $\mathbb{C}$
-Often one puts on a growth condition at cusp.

10. TOPICS

Goal: Classify generators of the algebra of holomorphic modular form. $M(\Gamma)$ for $\Gamma = PSL(2,\mathbb{Z})$ and dimension $M_{2k}(\Gamma)$ for $k \geq 0$.(Serre, course in Arithmetic.)
Prelimary remarks on cusps and holomorphic function at cusps.

10.1. **cusps.** Treat group $\Gamma \subseteq PSL(2,\mathbb{Z})$ of finite index.

Definition 10.1.1. Extend upper half plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, where $\mathbb{P}^1(\mathbb{Q})$ are the possible cusps.

Proposition 10.1.2. *Let $\Gamma \subseteq PSL(2,\mathbb{Z})$ be finite index. Then Gamma acts on $\mathbb{P}^1(\mathbb{Q})$ by linear fractional transformation.*

1. *$SL(2,\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ and the fixed transformation are $M = \pm I$.*
2. *Any finite index Γ has finitely many orbits on $\mathbb{P}^1(\mathbb{Q})$ of number $\leq [\Gamma(1) : \Gamma]$ (Orbits are called cusps and compactify $\mathbb{H}/\Gamma \cup \{cusp\}$).*

Proof. Check the action on $\mathbb{P}^1(\mathbb{Q})$. For this find $M \in SL(2,\mathbb{Z})$ that sends r to ∞, then the action is acing transitively.

Now write given Γ. $\Gamma(1) = \cup \Gamma a_i$, each coset gives an orbit Γa_i, each gives $[\Gamma(1) : \Gamma]$ orbits, apriori could be fewer of some. □

Proposition 10.1.3. *Given $\Gamma \subseteq \Gamma(1)$ finite index. For each $x \in \mathbb{P}^1(\mathbb{Q})$ in a cusp, if there is a subgroup Γ_x of Γ that stablizes the point $x \in \{cusp\}$, $\Gamma_x = \{M : Mx = x\}$, there is g that takes $g(\infty) = 0$, now we see that $g^{-1}\Gamma_x g = \{\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)^k\}$, where w is called the width of the cusp and $1 \leq w \leq [\Gamma(1) : \Gamma]$.*

Remark 10.1.4. Such Γ_x are called parabolic subgroup of Γ if under conjugate it is $\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right)$. The FLT has only one fixed point ∞.

Remark 10.1.5. Classify $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ as

$$\text{elliptic} \leftrightarrow \text{Two complex fixed point} \leftrightarrow |Tr| < 2$$
$$\text{parabolic} \leftrightarrow \text{One real fixed point} \leftrightarrow |Tr| = 2$$
$$\text{hyperbolic} \leftrightarrow \text{two real fixed point} \leftrightarrow |Tr| > 2$$

Proof. Check Shimura "Intro to arith theory of auto forms" Page 7. □

Example 10.1.6. $\Gamma(1)$ has one cusp and with width one.

Example 10.1.7. $\Gamma(2)$ has 3 cusps, 0, 1, ∞.
$[\Gamma(1):\Gamma(2)] = 6$, and it is a normal subgroup of $\Gamma(1)$ with $\Gamma(1)/\Gamma(2) \simeq PSL(\mathbb{Z}/2\mathbb{Z})$.
Note, all cusps of a normal subgroup of $\Gamma(1)$ has same width.

Example 10.1.8. (Theta group Γ_θ).
Here Γ_θ has two cusps 1 and ∞, with width 1 and 2, respectively.

Proposition 10.1.9.

a. $\Gamma(2) \subseteq \Gamma_\theta \subseteq \Gamma(1)$.
b. Γ_θ *is not a normal subgroup of* $\Gamma(1)$.
c. $\Gamma_\theta = < \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) >$
d. Γ_θ *is conjugate inside* $\Gamma(1)$ *to* $\Gamma_0(2)$ *and also to* $\Gamma^0(2)$.

Proof. take $g = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ for b. □

10.2. **Modular form at cusp.** Want a notion of "holomorphic" at cusps. Can compactify open Riemann orbifold $\mathbb{H}/\Gamma$ by gluing cusps.

To get a holomorphic structure, have to correct the surface at elliptic points, and correct at cusps with exponential change of vanishing, take $q = e^{2\pi i\tau/w}$, where w is the width of the cusp.

Definition 10.2.1. If $\left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. Have fourier expansion of function $F(\tau)$ in $\mathbb{H}/\Gamma$ at the cusp ∞.

$F(\tau) = \sum_{-\infty}^{\infty} a_n q^{n/w}$ where $q = e^{2\pi i\tau}$. Then $F(\tau)$ is meromorphic at cusp if $a_n = 0$ for all $n \leq -N_0$, holomorphic if $a_n = 0$ for $n \leq -1$, cuspidal if $a_n = 0$ for $n \leq 0$.

Definition 10.2.2. A weak modular form is holomorphic if holomorphic at all cusps and also inside $\mathbb{H}$.

A weak modular form is meromorphic if meromorphic at all cusps and also inside $\mathbb{H}$.

Definition 10.2.3. A holomorphic modular form $f(\tau)$ is cusp form if it is cuspidal at all cusps of Γ (This condition make the corresponding L-function has Euler product.)

10.3. **Algebra of Modular form.** $\Gamma \subseteq PSL(2,\mathbb{Z})$ finite index.

Definition 10.3.1. $M_{2k}(\Gamma)$= holomorphic modular form with weight $2k$.
$S_{2k}(\Gamma) := M^0_{2k}$ = cusp form with weight $2k$ of Γ.

Remark 10.3.2. Weight gives a grading on modular forms.
$M_k(\Gamma)M_L(\Gamma) \subseteq M_{k+l}(\Gamma)$ $S_k(\Gamma)S_L(\Gamma) \subseteq S_{k+l}(\Gamma)$

Theorem 10.3.3. *so $M(\Gamma) = \oplus M_{2k}(\Gamma)$ is a graded algebra over $\mathbb{C}$.*
$M_0(\Gamma)$ = holomorphic modular form of weight $0 = \mathbb{C} \cdot 1$
and cusp form
$S(\Gamma) \oplus S_{2k}(\Gamma)$ is an ideal of $M(\Gamma)$

10.4. **Finite dimensionality of space $M_{2k}(\Gamma)$.** Goal:
1.$M(\Gamma)$ is a ring over $\mathbb{C}$ and is generated by holomorphic Eisenstein series $G_4(\tau)$ and $G_6(\tau)$.
2. $\dim(M_{2k}(\Gamma(1)))$ is finite and $= \begin{cases} [k/6] & k \equiv 1 \mod 6 \\ 1 + [k/6] & else \end{cases}$
Minimal cusp form has weight 12 and $c\Delta(\tau) = q\prod(1-q^n)^24 = \eta(\tau)^24$.
Need to count order of zero and pole of holomorphic modular function $f(\tau)$.
Convention: at point $\tau \in \mathbb{H}$, $ord_\tau(f(\tau)) \equiv \mathrm{i}, \rho, \infty \mod \Gamma$.

Definition 10.4.1. Order at $\tau = \mathrm{i}\infty$. Set $\nu_\infty(f(\tau)) = ord_{q=0}(f(q))$.

Then we have the order $f(\tau)$ at an elliptic fixed point τ of order j, counted with weight $m_\tau = 1/j$ Then $\nu_\rho(f(\tau)) = 1/3$ (order of zero or pole of $f(\tau)$), $\nu_\mathrm{i}(f(\tau)) = 1/2$ (order of zero or pole of $f(\tau)$)
($\rho = e^{2\pi\mathrm{i}/3}$)

Proposition 10.4.2. *Let $f(\tau)$ be meromorphic modular form with weight $2k$, Then*

$$\sum_{regular\ \tau} \nu_\tau(f) + \nu_\infty(f) + \frac{\nu_\mathrm{i}(f)}{2} + \frac{\nu_\rho(f)}{3} = k/3.$$

(

$$\sum_{\tau\in\mathbb{H}^*} \frac{\nu_\tau(f)}{m_\tau} = \frac{k}{6}$$

in the second class)
$\nu_\tau(f)$ = order of zero or pole at $z = \tau$.
Check: GTM 7: Serre, Course of Authentic, Chapter 3, P85.

Proof. Integrate $\frac{1}{2\pi\mathrm{i}}\frac{f'}{f}(\tau)$ over a path on the modular surface, which basically a closed path along the side but avoiding the cusps.
Suppose f has no zero or pole on the vertical line $\Re(\tau) = \pm 1/2$, $|\tau| > 1$.
Method 1: Integration $= \sum_{\text{all zeros and poles inside contoure with multiplicity}} \nu_p(f)$, which excludes the zeros and poles excluding ∞, τ, ρ.
Method 2:
integration of top $= -\nu_\infty(f)$ by change variable to $q = e^{2\pi\mathrm{i}\tau}$ and the minus sign due to the orientation.
integration of left and integration of right cancels.
(Side note: Holomorphic(conformal) functions on $\mathbb{H}$ are contracting with respect to hyperbolic area. Moreover, either they are strictly contracting or they are hyperbolic isometries.)
integration of bottom left is $\frac{-1}{6}\nu_\rho(f)$
integration of bottom right is $\frac{-1}{6}\nu_{\rho'}(f) = \frac{-1}{6}\nu_\rho(f)$ since it is modular form.
Integration of middle cusp i is $-\frac{1}{2}\nu_\mathrm{i}(f)$
Modular form of weight k: Arc of bot left and arc of bot right are related under the action $S : \tau \mapsto 1/-\tau$, so they cancelled out except weight $2k$ of modular form has a contribution.

$\frac{df(s\tau)}{f(s\tau)} = \frac{df(\tau)}{f(\tau)} + 2k\frac{d\tau}{\tau}$.

Thus integration on the bottom arc is $\frac{k}{6}$.

Now, adding all the parts up, we have the proof of the proposition. □

Proposition 10.4.3. *Dimension of holomorphic form.*

(1) $M_{2k}(\Gamma(1)) = 0$ *if* $k \leq 0$ *and if* $k = 1$.
(2) $\dim(M_{2k}(\Gamma(1))) = 1$ *for* $k = 0, 2, 3, 4, 5$ *with basis* $1, G_4, G_6, G_4^2, G_4G_5$ *respectively.*
(3) *(Weight* 12 *form* Δ *is a cusp form).* $\Delta = g_x^3 - 27g_3^2 = (60G_4)^3 - 27(140G_6)^2$, *and multiplication by* Δ *give an isomorphism* $M_{2k-12}(\Gamma(1)) = S_{2k}(\Gamma(1)) = M_{2k}^0(\Gamma(1)) = \begin{cases} 0 & k \leq 5 \\ 1 & k = 6. \end{cases}$

Proof. Use the last proposition and basic number theory, we have 1.

Moreover,

Weight 0 case, it can only be constant function.

Weight 4 case, must have zero at ρ.

Weight 6 case, must have zero at i.

Weight 8 case, must have double zero at ρ.

Weight 10 case, must have zero at i and zero at ρ. Weight 12 case, there is Δ, which is a linear combination of G_4 and G_6, that is a cusp form, Note that multiplicaiton of the cusp form increase the multiplicity of the zero at infinity by 1, and thus it is an isomorphism. □

Corollary 10.4.4. $\dim(M_{2k}(\Gamma(1)))$ *is finite and* $= \begin{cases} [k/6] & k \equiv 1 \mod 6 \\ 1 + [k/6] & \textit{else} \end{cases}$

Proof. By induction. □

Proposition 10.4.5. *The space* M_k *has for basis the family of monomials* $G_4^{\alpha}G_6^{\beta}$ *and* $2\alpha + 3\beta = k$ *if* α *and* β *are nonnegative.*

And we have that $M(\Gamma(1)) \simeq \mathbb{C}[x, y]$, $x = G_4(\tau)$, $y = G_6(\tau)$.

11. Delta function

$\Delta(\tau)$ as before.

$\Delta(\tau) = (2\pi)^{12}q(\sum_{h=1}^{\infty}\tau(n)q^n) = (2\pi)^{12}q(1 - 24q + 252q^2 + \ldots) = (2\pi)^{12}q\prod_{r=1}^{\infty}(1-q^r)^{24}$.

Ramanujan conjecture that (1916):

(1) Multiplicity
(2) Formula for prime powers in terms of $\tau(p)$

    These are established soon after Mordell related to Hecke correspondence symmetry.
(3) Ramanujan conjecture $|\tau(p)| < 2p^{11/2}$, parallel to the function field RH.

There is a Fourier coefficient bound for the cusp form. $|a(n)| \leq\leq n^K$ (Hecke).

Note that 3 is proved by Deligne from RH on the AV on function field.

11.1. Modular invariant $j(\tau)$.

$j(\tau) = 1728J(\tau) = 1728\frac{g_2^3(\tau)}{\Delta(\tau)}$.

Have $J(\tau)$ to be weight 0 meromorphic modular function.
Moreover,
$j(\tau) = \sum_{n=-1}^{\infty} q^n$ has a simple pole at $\tau = \infty$ and is $\frac{1}{q} + 744 + 196884q + 21493760q^2$.

Theorem 11.1.1. • *The function j is a meromorphic modular form at 0 on $\mathbb{H}/\Gamma(1)$.*

- *It is holomorphic in $\mathbb{H}$ and has a single pole at the cusp $\mathrm{i}\infty$.*
- *It defines a bijection $\mathbb{H}/\Gamma(1) \simeq \mathbb{C}$.*

Proof. (1) Note that G_4 is non-cusp form, Δ is cusp form, thus done.
(3) Check, know that $j(\tau)$ has a simple pole at infinity, and has exactly one zero by the valuation argument. Then we can see that $j(\tau)$ take any value in $\mathbb{C}$ exactly once. □

Proposition 11.1.2. *("$j(\tau)$ generates all the meromorphic modular form of $\mathbb{H}^*/\Gamma(1)$")*
TFAE:

(1) *$f(\tau)$ is meromorphic function of weight 0 for $\Gamma(1)$.*
(2) *$f(\tau)$ is the quotient of two holomorphic modular forms of equal weight.*
(3) *$f(\tau)$ is a rational function of $j(\tau)$*

Proof. Ignored. □

11.2. **Numerology of $j(\tau)$.** $\frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots$.
And we have that $j(\tau)$ has the terms $c(1) = 196884 = 2^3 \cdot 3^3 \cdot 1823 = 196883 + 1$, $c(2) = 1 + 196883 + 21296876$ coefficients grows in a rate $\exp(c\sqrt{n})$.
D.H.Lehmer showed:

- $n \equiv 0(\mod 2^a)$ then $c(n) \equiv 0 \mod 2^{3a+8}$
- $n \equiv 0(\mod 3^a)$ then $c(n) \equiv 0 \mod 3^{2a+3}$
- $n \equiv 0(\mod 5^a)$ then $c(n) \equiv 0 \mod 5^{a+1}$
- $n \equiv 0(\mod 7^a)$ then $c(n) \equiv 0 \mod 7^a$
- $n \equiv 0(\mod 11^a)$ then $c(n) \equiv 0 \mod 11^a$

John Mckay observed that coincidence with the character table of the monstrous group $M = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$.
Conjecture: $c(n)$ is the degree pf the irreducible representation of monster group.
Numerology that Jeff has: $196883 - 1 = 2 \cdot 7^4 \cdot 41$, $21296876 - 1 = 5^{\cdot}29 \cdot 47$.

11.3. **Coefficients at holomorphic modular form.** $f(q) = \sum_{n=0} a_n q^n$ a weight $2k$ holomorphic form.

Proposition 11.3.1. *If $f(\tau) = G_{2k}(\tau)$ is Eisenstein series of $2k$, then the function coefficients a_n has that $a_n = O(n^{2k-1})$. Have $An^{2k-1} \le |a_n| \le Bn^{2k-1}$.*

Proof. Note that $a_n = \pm A_k(\sigma_{2k-1}(n))$. $\sigma_s(n) = \sum_{d|n} d^s$.
Then we know that $c \cdot n^{2k-1} \ge \sigma(n) \ge n^{2k-1}$, and $c \le \sum m \in \mathbb{N} \frac{1}{m^{2k-1}} \le \phi(2k-1)$ □

Theorem 11.3.2. *(Hecke).*
If $f(\tau)$ is a holomorphic cusp form of weight $2k$ on $\Gamma(1)$, then we have that $a_n = O(n^k)$.

Proof. Here $a_0 = 0$ since cusp. $f(\tau) = O(q) = O(e^{-2\pi \mathrm{i}\tau})$. decreases
Consider nonholomorphic function $\psi(\tau) = |f(\tau)|y^k$. This function is invariant under $\Gamma(1)$, and thus is bounded on $\mathbb{H} \cup \{cusp\}$, call M the bound.
We then integrate by Cauchy $a_n = \frac{1}{2\pi \mathrm{i}} \int f(q) q^{-n} \frac{dq}{q}$. Then $|a_n| \le e^{-2\pi n y} M y^{-K}$. Pick $y = 1/n$. Done. □