NOTES ON TATE'S THEOREM

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ABSTRACT. This is the note for the talk about the Tate's theorem for the seminar on the local class field theory seminar.

Contents

1. Background

Definition 1.1. Define the induced module and coinduced module for $H \subseteq G$ and denote them as:

$$\mathrm{Ind}_H^G(X) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} X$$
$$\mathrm{CoInd}_H^G(X) = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], X)$$

Define the G-induced module and G-coinduced module and denote them as:

$$\mathrm{Ind}^G(X) := \mathbb{Z}[G] \otimes X$$

$$\mathrm{CoInd}^G(X) = \mathrm{Hom}(\mathbb{Z}[G], X)$$

Also, we have a proposition and a corollary for the induced and coinduced module.

Proposition 1.2. *Suppose that H is a subgroup of finite index in G and B is a H-module. Then we have a canonical isomorphism of G-modules:*

$$\chi : \mathrm{CoInd}_H^G(B) \simeq \mathrm{Ind}_H^G(B)$$
$$\varphi \mapsto \sum_{\overline{g} \in H \backslash G} g^{-1} \otimes \varphi(g)$$

where for each $\overline{g} \in H\backslash G$, the element $g \in G$ is an arbitrary choice of representative of g.

Corollary 1.3. *For G-modules A we have the exact sequence*

$$0 \to A \xrightarrow{\iota} \mathrm{CoInd}^G(A) \to A^1 \to 0$$

$$0 \to A^{-1} \to \mathrm{Ind}^G(A) \xrightarrow{\pi} A \to 0$$

where $A^1 \simeq J_G \otimes A \simeq \mathrm{Hom}_{\mathbb{Z}}(I_G, A)$, $A^{-1} \simeq I_G \otimes_Z A$

Moreover, we also knew from the previous talk that for Tate Cohomology,

$$H^{-2}(G,\mathbb{Z}) = G^{ab}$$
$$H^{-1}(G,A) = A[N_G]/I_G A$$
$$H^0(G,A) = A^G/N_G A$$
$$H^1(G,A) = \{\text{Cross homomorphism}\}/\{x : G \to A | x(\sigma) = \sigma a - a\}$$

The object of this talk:

Theorem 1.4 (Tate's theorem)**.**
For finite group G, and $\alpha \in H^2(G,A)$. Suppose that for every p, we have that $H^1(G_p,A)$ trivial and $H^2(G_p,A)$ is cyclic of order $|G_p|$ generated by the restriction of α.

Then the map:

$$H^i(H,\mathbb{Z}) \to H^{i+2}(H,A)$$
$$\beta \mapsto Res(\alpha) \cup \beta$$

are isomorphism for all $i \in \mathbb{Z}$ and subgroup H of G.

Now, note that given a class formation A, i.e. a G-module such that $H^1(G,A) = 0$ for all $H \subseteq G$, if further we consider $G = Gal(L/K)$, then after checking necessary condition, from the Tate's theorem, we will get

$$G^{ab}_{L|K} = H^{-2}(G,\mathbb{Z}) \simeq H^0(G,A) \simeq A_K/N_{L|K}A_L$$

.

2. Cohomological Triviality

Before we talk, we need a theorem about cohomological triviality.

Definition 2.1. A G-module A is said to be cohomological trivial if $H^i(H,A) = 0$, $\forall H \subseteq G$, $\forall i \in \mathbb{Z}$.

Example 2.2. (1) Induced G-modules are cohomologically trivial.
(2) Projective G-modules are cohomologically trivial.
Proof: since if P is projective, then we have that $H^i(H,P) \hookrightarrow H^i(P) \oplus H^i(Q) \simeq H^i(P \oplus Q)$.

2.1. Tate cohomology of Cyclic group.

Theorem 2.3. *Given G to be the cyclic group and A be a G-module, then we have $H^q(G,A) \simeq H^{q+2}(G,A)$.*

Note that in this case we actually have $\mathbb{Z}[G] = \oplus^n \mathbb{Z}\sigma^i$, $N_G = 1+\sigma+\ldots+\sigma^{n-1}$, $I_G = \mathbb{Z}[G](\sigma-1)$

Proof. We just need to show that $H^{-1}(G,A) \simeq H^1(G,A)$, since the other relations can be done by dimension shifting.

Now, consider $x \in Z^1$, then we have that

$$x(\sigma^k) = \sigma x(\sigma^{k-1}) + x(\sigma) = \ldots = \sum_{i=1}^{k-1} \sigma^i x(\sigma)$$

and

$$x(1) = 0.$$

but then since

$$N(G)(x(\sigma)) = \sum_{i=1}^{n-1} \sigma^i x(\sigma) = x(\sigma^n) = x(1) = 0.$$

Therefore, $x(\sigma) \in A[N_G]$.

Therefore, we have that $x \mapsto x(\sigma)$ is an isomorphism on the cocycle.

Moreover, given any $x \in B^1$, we have that

$$\begin{aligned} x \in B^1 &\Leftrightarrow x(\sigma^k) = \sigma^k a - a \\ &\Leftrightarrow x(\sigma) = \sigma a - a \\ &\Leftrightarrow x(\sigma) \in I_G A = B^{-1} \end{aligned}$$

Therefore, we have that for cyclic group G, the desired properties holds, and furthermore,

$$H^{2q} \simeq (G, A) \simeq H^0(G, A)$$
$$H^{2n+1}(G, A) \simeq H^1(G, A)$$

□

2.2. **Cohomological triviality.** Here I basically follows Sharifi's note on Group cohomology, on the part about cohomological triviality and we denote the G-invariant of A to be A^G, and the G-coinvariant of A to be $A_G \simeq A/I_G A$

Lemma 2.4. *Suppose that G is a p-group and A is a G-module of exponent dividing p. Then $A = 0$ if and only if either $A^G = 0$ or $A_G = 0$.*

Proof. If $A^G = 0$, and let $a \in A$. Then $B := < a > \subseteq A$ is finite, and $B^G = 0$. Thus the G-orbits in B are either 0 or have order a multiple of p. Since B has p-power order, the order has to be 1, so $B = 0$. Since the choice of a was arbitrary, $A = 0$. On the other hand, if $A_G = 0$, then $X = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{F}_p)$ satisfies $pX = 0$ and $X^G = \mathrm{Hom}_{\mathbb{Z}[G]}(A_G, \mathbb{F}_p) = 0$ and thus $X = 0$. □

Lemma 2.5. *Suppose that G is a p-group and that A is a G-module of exponent dividing p. If $H^{-2}(G, A) = 0$, then A is free as an $\mathbb{F}_p[G]$-module.*

Proof. Lift an $\mathbb{F}_p$-basis of A_G to a subset Σ of A.

For $B := < \Sigma > \subseteq A$ generated by Σ, the quotient A/B has trivial G-invariant group, hence is trivial by the above lemma. Thus we have that $< \Sigma >$ generates A as an $\mathbb{F}_p[G]$-module. Now, if we let F be the free $F_p[G]$-module generated by Σ, we then have a canonical surjection $\pi : F \to A$, and we let R be the kernel. Since we have that $H^{-2}(G, A) = 0$ thus we have the exact sequence

$$0 \to R_G \to F_G \xrightarrow{\hat{\pi}} A_G \to 0$$

By definition, we have that $\hat{\pi}$ is an isomorphism. Therefore, we have that $R_G = 0$, thus $pR = 0$ and by the above lemma $R = 0$, thus π is also isomorphism. □

We can use this to prove a proposition as following.

Proposition 2.6. *Suppose that G is a p-group and that A is a G-module of exponent dividing p. The following are equivalent:*

(i) *A is cohomologically trivial*
(ii) *A is a free $\mathbb{F}_p[G]$-module.*
(iii) *There exists $i \in Z$ such that $H^i(G, A) = 0$*

Proof. (i) proves (iii) is trivial.

(ii) proves (i) is almost immediate. If we have that A is $\mathbb{F}_p$ free with basis I, then

$$A \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}} (\oplus_{i \in I} \mathbb{F}_p)$$

and thus A is an induced module, thus cohomologically trivial.

(iii) proves (ii) is also not hard. Firstly, notice that the module after dimension shifting is going to be killed by p since A is killed by p. By dimension shifting, we have that $H^i(G, A) \simeq H^{-2}(G, A^{2+i}) = 0$, but then we know by the last lemma that A^{2+i} is also cohomologically trivial, thus, we have that A is also cohomologically trivial. And thus by the above lemma, we are done. □

Now we can use this to prove one of the key proposition.

Proposition 2.7. *Suppose that G is a p-group and A is a G-module with no elements of order p. The following are equivalent:*

(1) *A is cohomologically trivial.*
(2) *There exists $i \in Z$ such that $H^i(G, A) = H^{i+1}(G, A) = 0$.*
(3) *A/pA is free over $\mathbb{F}_p[G]$.*

Proof. (1) implies (2) is trivial.

(2) implies (3) is also not hard. Since A has no p-torsion,

$$0 \to A \xrightarrow{p} A \to A/pA \to 0$$

is exact. By (2) and l.e.s. in Tate cohomology, we have $H^i(G, A/pA) = 0$. By previous proposition, A/pA is free over $\mathbb{F}_p[G]$.

Now we prove (3) implies (1). By previous proposition, A/pA is cohomologically trivial, and therefore multiplication by p is an isomorphism on each $H^i(H, A)$ for each subgroup H of G for every $i \in Z$. However, the latter cohomology groups are annihilated by the order of H, so must be trivial since H is a p-group. □

Now we are actually pretty close to the result we want. However, to get the final theorem we want, there are two more preliminary lemmas.

Lemma 2.8. *Suppose that G is a p-group and A is a G-module that is free as an abelian group and cohomologically trivial. For any G-module B which is p-torsion free, we have that $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ is cohomologically trivial.*

Proof. Since B has no p-torsion and A is free over $\mathbb{Z}$, we have the following exact sequence:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(A, B) \xrightarrow{p} \mathrm{Hom}_{\mathbb{Z}}(A, B) \to \mathrm{Hom}_{\mathbb{Z}}(A, B/pB) \to 0$$

and moreover, $\mathrm{Hom}_{\mathbb{Z}}(A, B)$ has no p-torsion and

$$\mathrm{Hom}_{\mathbb{Z}}(A/pA.B/pB) \simeq \mathrm{Hom}_{\mathbb{Z}}(A, B/pB) \simeq \mathrm{Hom}_{\mathbb{Z}}(A, B)/p\,\mathrm{Hom}_{\mathbb{Z}}(A, B)$$

Since A/pA is free over $\mathbb{F}_p[G]$ and if we denote the index set to be I, we have

$$\mathrm{Hom}_{\mathbb{Z}}(A/pA.B/pB) \simeq \prod_{i\in I} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{F}_p[G], B/pB) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \prod_{i\in I} B/qB)$$

Therefore we see that $\mathrm{Hom}_{\mathbb{Z}}(A, B/pB)$ is G-coinduced and thus is also free over $\mathbb{F}_p[G]$. Now by the previous proposition, we have that $\mathrm{Hom}(A, B)$ is cohomologically trivial.

□

Proposition 2.9. *Let G be a finite group and A a G-module that is free as an abelian group. Then A is cohomologically trivial if and only if A is a projective G-module.*

Proof. In the example of cohomological trivial G-module, we already see that if A is a projective G-module, then we will have that A is cohomologically trivial.

Thus here we just need to prove the converse statement is true. Note that A is free G-module, thus from the definition we have that $\mathrm{Ind}^G(A)$ is also free. Thus we obtained an exact sequence as the following:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(A, A^{-1}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Ind}^G A) \to \mathrm{Hom}_{\mathbb{Z}}(A, A) \to 0$$

Then by the last proposition we have that $\mathrm{Hom}_{\mathbb{Z}}(A, A^{-1})$ is cohomologically trivial, and thus by the long exact sequence of the exact sequence we have that $\mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Ind}^G(A)) \to \mathrm{Hom}_{\mathbb{Z}}(A, A)$ is surjective, and thus the identity map lifts to a map from A to $\mathrm{Ind}^G(A)$ and thus we have that A is projective as a G-module. □

Proposition 2.10. *G be a finite group and $\forall p$, G_p Sylow subgroup of G, if A is cohomologically trivial as a G-module if and only if it is cohomologically trivial as G_p-module $\forall p$.*

Proof. (proof of the proposition) Consider now A is cohomologically trivial, then $\forall G_p$, H be a subgroup of G, any Sylow p-subgroup H_p of H contained in gG_pg^{-1}, a conjugation of G_p by cohomologically trivial of G_p. Thus we have that $H^i(g^{-1}G_pg, A) = 0$. As g^* is isomorphism, thus we have that $H^i(H_p, A) = 0$.

Therefore, we have that $Res : H^i(H, A) \to H^i(H_p, A) = 0$, $\forall p$, and thus $H^i(H, A) = 0$ by one of the lemma Alex proved. □

Now, we want to prove a theorem about cohomological triviality.

Theorem 2.11. *G be a finite group and A be a G-module. Then the following are equivalent:*

1. *A is cohomologically trivial.*
2. *For each prime p, there exist $i \in \mathbb{Z}$ such that $H^i(G_p, A) = H^{i+1}(G_p, A) = 0$.*
3. *$\exists$ an exact sequence of G-modules such that*

$$0 \to P_1 \to P_0 \to A \to 0$$

where P_1 and P_0 are projective.

Proof. Note that one way is trivial. Since If A is cohomologically trivial, then we have (2) automatically.

Now suppose (2), we want to show (3). Let F be a free G-module that surjects onto A (Since every module is isomorphic to a free quotient), and let R be the kernel. As F is cohomologically trivial, thus by the long exact sequence we have $H^{j1}(G_p, A) \simeq H^j(G_p, R)$

for every $j \in \mathbb{Z}$. It follows that $H^j(G_p, R)$ vanishes for two consecutive values of j. Since R is $\mathbb{Z}$-free as it is a subgroup of F, we have by by the above propositions that R is projective.

(3) implies (1) follows from the fact that projective modules are cohomologically trivial and the long exact sequence.

□

3. Tate's theorem

Now we have the desired statement in cohomological triviality, we can show the following proposition.

Proposition 3.1. *We have $\psi : A \to B$ as an G-module homomorphism and it can be viewed as G_p-module homomorphism, denoted as ψ_p. Suppose $\forall p$, $\exists j \in \mathbb{Z}$ such that*

$$\psi_p^* : H^i(G_p, A) \to H^i(G_p, B)$$

is surjective for $i = j - 1$, isomorphism for $i = j$, and injective for $i = j + 1$ Then we have

$$\psi^* : H^i(H, A) \to H^i(H, A)$$

is isomorphism $\forall i \in \mathbb{Z}$ and $\forall H \subseteq G$.

Proof. Consider the map $\psi \oplus \tau : A \to B \oplus \mathrm{CoInd}^G(A)$ as a canonical injection, and we let C be its cokernel.

Note that $H^i(B \oplus \mathrm{CoInd}^G(A)) \simeq H^i(B)$, thus we have the long exact sequence

$$\dots H^j(G_p, A) \xrightarrow{\psi_p^*} H^j(G_p, B) \to H^j(G_p, C) \xrightarrow{\delta} H^{j+1}(G_p, A) \xrightarrow{\psi_p^*} H^{j+1}(G_p, B) \to \dots$$

Now if we consider $j = i - 1$, since it is surjective on H^{i-1} and is isomorphic on H^i, thus we have $H^i(G_p, C) = 0$.

Similarly, if we take $j = i$, we have that $H^{i+1}(G_p, C) = 0$.

Therefore, by the theorem above, we have that C is cohomologically trivial and ψ^* is then isomorphism. □

Now using the proposition we can prove the main theorem of today's talk.

Theorem 3.2 (Main Theorem)**.**

A, B, C are G-modules and $\theta : A \otimes_{\mathbb{Z}} B \to \mathbb{C}$ is a G-module map. For some $k \in \mathbb{Z}$, we have $\alpha \in H^k(G, A)$, $\forall H \in G$,

$$\Theta^i_{H,a} : H^i(H, B) \to H^{i+k}(H, C)$$

, where

$$\theta^i_{H,a}(\beta) = \theta^*(Res(\alpha) \cup \beta)$$

and for all p prime, there exists $j \in \mathbb{Z}$ such that $\theta^*_{G_p,a}$ is surjective for $i = j - 1$, isomorphic for $i = j$, and is injective for $i = j - 1$. Then $\forall H \subseteq G$ and $i \in \mathbb{Z}$, $\theta^i_{H,\alpha}$ is an isomorphism.

Proof. Using the argument of dimension shifting of A, we can reduce the condition so that we just consider $k = 0$, $\psi : B \to C$ by $\psi(b) = \theta(a \otimes b)$, where $a \in A^G$ represents α.

First we can check that ψ is well defined and is a map of G-module.

$$\psi(gb) = \theta(a \otimes gb) = \theta(ga \otimes gb) = g\theta(a \otimes b) = g\psi(b)$$

Now we want, for every degree i, $\psi^*(H^i)(H, B) \to H^i(H, C)$ agrees with $\theta^*(Res(\alpha) \cup \beta)$.

We will do a two side induction here to prove the claim.

Base step: For degree 0, we know that psi^* is induced by $\psi : B^H \to C^H$, where $b \mapsto \theta(a \otimes b)$.

Now we consider one side first.

Note that we have the short exact sequence

$$0 \to A^{-1} \to \mathrm{Ind}^G(A) \xrightarrow{\pi} A \to 0$$

And we have the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes_{\mathbb{Z}} B^{-1} & \longrightarrow & A \otimes_{\mathbb{Z}} \mathrm{Ind}^G(B) & \longrightarrow & A \otimes_{\mathbb{Z}} B & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow & & \downarrow \\
0 & \longrightarrow & (A \otimes_{\mathbb{Z}} B)^{-1} & \longrightarrow & \mathrm{Ind}^G(A \otimes_{\mathbb{Z}} B) & \longrightarrow & A \otimes_{\mathbb{Z}} B & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \mathrm{Ind}^G(\theta)} & & \downarrow{\scriptstyle \theta} & & \downarrow \\
0 & \longrightarrow & C^{-1} & \longrightarrow & \mathrm{Ind}^G(C) & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

And we have a map $\psi' : B^{-1} \to C^{-1}$ by mapping $b' \mapsto \theta'(a \otimes b')$ for all $b' \in B^{-1}$.

We then have the following two commuting diagram where one is induced by the ψ' and ψ, and the other is from the $\Theta^i_{H,\alpha}$ as in the question.

$$\begin{array}{ccccc}
H^{i-1}(H, B) & \xrightarrow{\simeq} & H^i(H, B^{-1}) & \qquad & \beta' \\
\downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \psi'^*} & & \downarrow \\
H^{i-1}(H, C) & \xrightarrow{\simeq} & H^i(H, C^{-1}) & & (\theta')(Res(\alpha) \cup \beta')
\end{array}$$

and

$$\begin{array}{ccccc}
H^{i-1}(H, B) & \xrightarrow{\simeq} & H^i(H, B^{-1}) & \qquad & \beta' \\
\downarrow{\scriptstyle \Theta^{i-1}_{H,\alpha}} & & \downarrow{\scriptstyle \Theta^{i}_{H,\alpha}} & & \downarrow \\
H^{i-1}(H, C) & \xrightarrow{\simeq} & H^i(H, C^{-1}) & & (\theta')(Res(\alpha) \cup \beta')
\end{array}$$

We get $\psi^* = \Theta^{i-1}_{H,\alpha}$

Now we consider the exact sequence for coinduced module, then we get the commutative diagram

$$\begin{CD}
0 @>>> A \otimes_{\mathbb{Z}} B @>>> \mathrm{CoInd}^G(A) \otimes_{\mathbb{Z}} B @>>> A \otimes_{\mathbb{Z}} B^1 @>>> 0 \\
@VVV @VVV @VV{\simeq}V @VV{\simeq}V @VVV \\
0 @>>> A \otimes_{\mathbb{Z}} B @>>> \mathrm{CoInd}^G(A \otimes_{\mathbb{Z}} B) @>>> (A \otimes_{\mathbb{Z}} B)^1 @>>> 0 \\
@VVV @VV{\theta}V @VV{\mathrm{CoInd}^G(\theta)}V @VV{\theta'}V @VVV \\
0 @>>> C @>>> \mathrm{Ind}^G(C) @>>> C^1 @>>> 0
\end{CD}$$

And using a similar statement, we can prove the argument $\psi^* = \Theta^{i+1}_{H,\alpha}$.

Now we consider a general $k \in \mathbb{Z}$.

Note that for $\alpha \in H^{k-1}(H, A)$, we have $\alpha' := \delta(\alpha) \in H^k(H, A^{-1})$.

Consider the following commutative diagram:

$$\begin{CD}
0 @>>> (A^{-1} \otimes_{\mathbb{Z}} B @>>> \mathrm{Ind}^G(A) \otimes_{\mathbb{Z}} B @>>> A \otimes_{\mathbb{Z}} B @>>> 0 \\
@VVV @VV{\theta'}V @VV{\mathrm{Ind}^G(\theta)}V @VV{\theta}V @VVV \\
0 @>>> C^{-1} @>>> \mathrm{Ind}^G(C) @>>> C @>>> 0
\end{CD}$$

Therefore, we see that $\Theta_{H,\alpha'} : H^i(H, B) \to H^{i+k}(H, C^{-1})$ is well defined and we have the following diagram commute:

$$\begin{CD}
H^i(H, B) @= H^i(H, B) \\
@VV{\Theta_{H,\alpha'}}V @VV{\Theta_{H,\alpha'}}V \\
H^{i+k-1}(H, C) @>{\delta}>{\simeq}> H^{i+k}(H, C^{-1})
\end{CD}$$

Now we just apply the last proposition to $\psi^* = \Theta^i_{G_p,\alpha}$, we have that since it is surjective for $i = j - 1$, isomorphic for $i = j$, and injective for $i = j + 1$, $\Theta^i_{H,\alpha}$ is an isomorphism for all $i \in \mathbb{Z}$ and for all $H \subseteq G$. ☐

Tate's famous theorem is followed then as a special case of the main theorem

Theorem 3.3. *Given $\alpha \in H^2(G, A)$, suppose $\forall p$, $H^1(G_p, A)$ is trivial and $H^2(G_p, A)$ is cyclic of order $|G_p|$ and is generated by the restriction of α. Then the map*

$$H^i(H, \mathbb{Z}) \to H^{i+2}(H, A)$$

where

$$\beta \mapsto Res(\alpha) \cup \beta$$

are isomorphism $\forall i \in \mathbb{Z}$ and subgroup $H \subseteq G$.

Proof. Consider $H = G_p$, and then the map is surjective for $i = -1$ since $H^1(G_p, A) = 0$, is isomorphic for $i = 0$ since $H^0(G_p, \mathbb{Z}) \simeq \mathbb{Z}/|G_p|\mathbb{Z} = H^2(G_p, A)$ by $n \mapsto nRes(\alpha)$, and injective for $i = 1$ since $H^1(G_p, A) = 0$. Therefore, we have the desired result. ☐

Note that we have $H^{-2}(G, \mathbb{Z}) = G^{ab}$, and thus in the good cases, if we take $G = Gal(L/K)$, we should have $G^{ab}_{L/K} \simeq H^0(G, A) = A_K/N_{L/K}A_L$.