Lattice Polygons

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Math Club Talk on Lattice Polygons

The URL for the Beamer Presentation: "Pick's Theorem and beyond" is available at
http://www-personal.umich.edu/~yiwchen/index.html

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Given any polygon, how do we find the area of the polygons?

Triangle Area Formula

If what we are having is a triangle, we have:

- The formula:

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$$A = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4)$$

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- Bretschneider's formula (Generalized Heron's formula):

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Pick's Theorem about Lattice Polygons

As we are in the case of general polygons (not necessarily convex), the fomulas mentioned above will gets very complicated.
However, there is a magic formula that allows us to compute the area of a certain type of polygon just by counting!

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Theorem (Pick's Theorem)

The area of a simple lattice polygon P is given by

$$A(P) = i + \frac{1}{2}b - 1 = l - \frac{1}{2}b - 1$$

where i, b and l are, respectively, the number of interior lattice points, the number of the boundary points, and the total number of the lattice points of the polygon P.

Who is Georg Pick?

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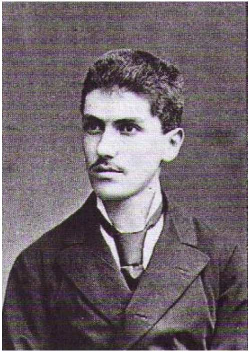

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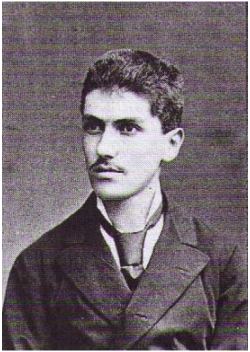
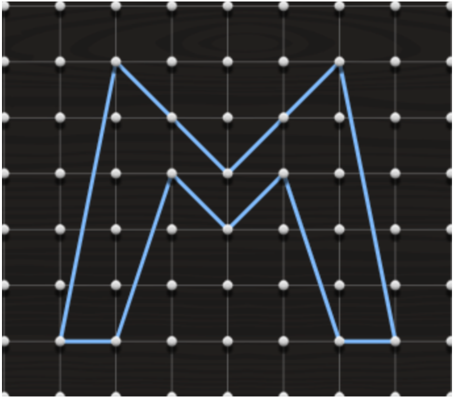


The Austrian Georg Pick completed his thesis at the University of Vienna under Königsberger and Weyr. Except for visiting Felix Klein in Leipzig in 1884, he worked his whole career at the Charles Ferdinand University in Prague. He returned to Vienna upon retirement in 1927. He died in the Theresienstadt Concentration Camp in 1942. Pick wrote papers in differential geometry and complex analysis. He headed the committee to appoint Albert Einstein to the chair of mathematical physics in 1911. He introduced Einstein to the recent work by Ricci-Curbastro and Levi-Civita in curved manifolds, without which Einstein couldn't have formulated his theory of General Relativity of curved spacetimes.

Examples: Pick's Theorem



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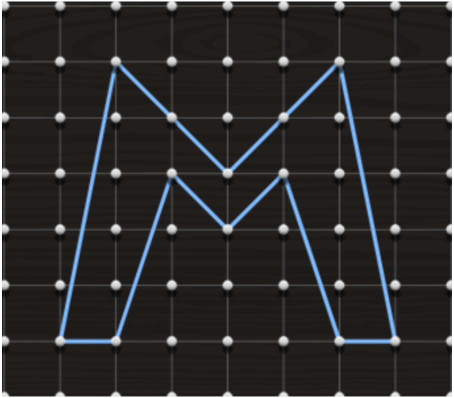
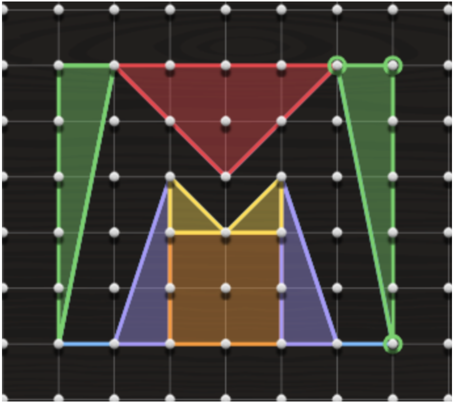


Here $I = 8,\ B = 12,\ A = I + \frac{1}{2}B - 1 = 8 + 6 - 1 = 13.$

Example: Geometric Computation of Area



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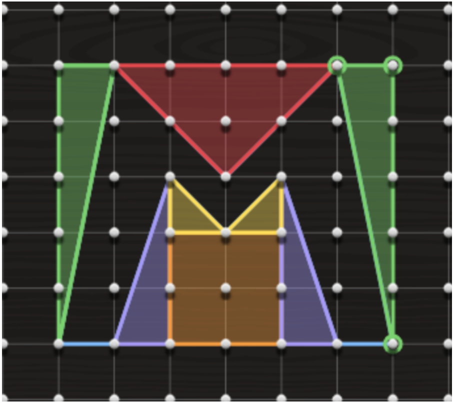
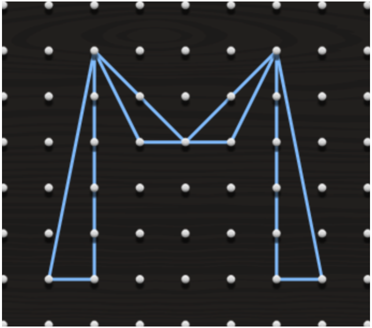


$$A = 30 - 5 - 4 - 1 - 3 - 4 = 13$$

In the case where the polygon is not simple



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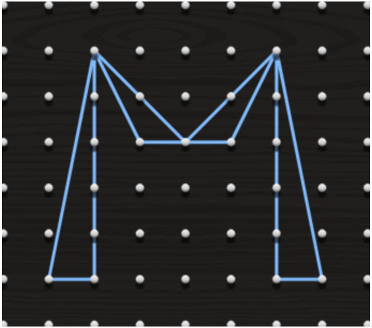

Here $I = 0$, $B = 19$, $A = I + \frac{1}{2}B - 1 = 0 + 19/2 - 1 = 8.5$.

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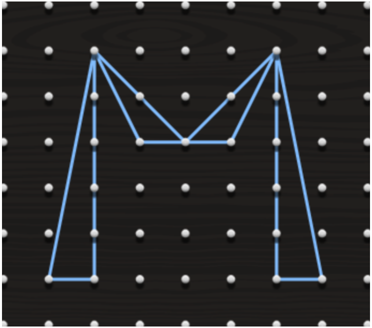

Here $I = 0$, $B = 19$, $A = I + \frac{1}{2}B - 1 = 0 + 19/2 - 1 = 8.5$.
But

$$A = 1 \times 5 \times 2/2 + 1 \times 2 \times 2/2 = 7.$$

Proof of the Pick's Theorem: Weight function

We will follow the proof of Varberg.

Definition

Given any polygon P, we can assign every single lattice point L_k a *weight* by $w_k = \frac{\theta_k}{2\pi}$ where θ_k is the "visiblity" angle for L_k sees into P.
We can then define the total weight $W(P) = \sum_{L_k \in P} w_k(L_k)$.

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Example

For interior lattice L_i, $w_k = 1$.
For the boundary lattice that is not a vertex, $w_k = 1/2$.
For the vertices which is a right angle corner point, $w_k = 1/4$.

Proof: Total weight equals to area.

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$W(P) = A(P).$

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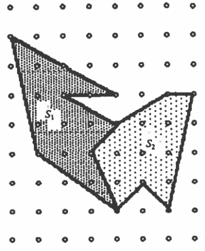

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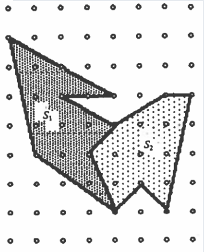
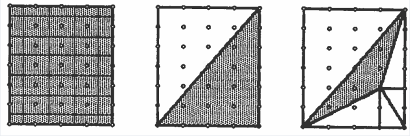


Firstly, W is additive, as the visibility angles of S_1 and S_2 adds up at the common lattice point to give the visibility angle of P as illustrated below.
Then, note that we can decompose any lattice polygon into a union of lattice triangles, thus if the lemma holds for lattice triangles, we are done.
Thus, we just need to prove that the lemma holds for lattice triangles.

□

Proof of the lemma: Triangle case.

Continued.



We will prove the lattice triangle case sucessively.
Firstly, we will show that for the lattice rectangles that are having the sides parallel to lattices. This is obvious.
Secondly, the case where a triangle having a right angle. This is done by using the first case divide by 2.
Thirdly, for any general lattice triangle, we can always find the embedded rectangle and subdivide the rectangle into right triangles along with our initial triangle, and then we use the additivity of both W and A. □

Proof of Pick's Theorem

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Proof.

A simple polygon will have the outer angle sum 2π, thus the sum of visible angles along the boundary is $b\pi - 2\pi$.
Therefore,

$$A(P) = W(P) = \sum_{L_k \in I} w_k(L_k) + \sum_{L_k \in B} w_k(L_k)$$
$$= i + \frac{(b-2)\pi}{2\pi} = i + \frac{1}{2}b - 1 = l = \frac{1}{2}b - 1.$$

Generalized Pick's Theorem

Definition (Euler Characteristic)

Euler Characteristic χ of a polygon is defined by

$$\chi = v - e + f,$$

Where v is the total number of the vertices, e is the total number of edges, and f is the total number of the faces, of the triangulation of P, heuristically, if the polygon has h holes, then

$$\chi = 1 - h.$$

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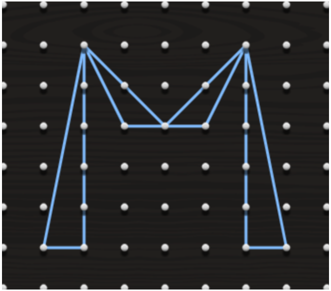
Theorem (General Pick's Theorem)

The area of a lattice polygon P (not necessarily simple) is given by

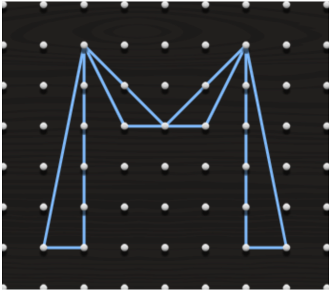
$$A(P) = v - \frac{1}{2}e_h - \chi,$$

where v is total number of lattice points, e_h is the number of edges on the boundary of P and that χ is the Euler characteristic of P.

Example revisited: Generalized Pick's Theorem



Example revisited: Generalized Pick's Theorem



Here $I = 0$, $B = 19$, $e_h = 22$, $\chi(P) = 1$. Therefore we have

$$A = 19 - \frac{1}{2} \times 22 - 1 = 19 - 22/2 - 1 = 7$$

. And recall our calculation,

$$A = 1 \times 5 \times 2/2 + 1 \times 2 \times 2/2 = 7.$$

Proof of the Generalized Pick's Theorem

We will prove it assuming primary triangulation of polygons (so we can decompose polygons into the triangles with area 1/2).
We denote v, e and f as the number of vertices, edges and faces of the decomposition, as noted above.
Since each triangle has 3 edges and each edge is shared by two triangles, we have $3f = 2e - e_h$.
Therefore, we have $f = 2e - 2f - e_h = 2v - e_h - 2\chi$, and

$$A(S) = \frac{1}{2}f = v - \frac{1}{2}e_h - \chi$$

Higher Dimensional Analogs

Reeve has given a formula for volume of three dimensional polyhedra involving also the Euler-Poincaré characteristic from algebraic topology. However, no formula involving only counts of lattice points on faces of P can exist. In fact, no such formula exists for lattice tetrahedra (the convex hull of four lattice pints in three dimensions).

Higher Dimensional Analog: Reeve tetrahedron

Theorem

For three dimensional lattice tetrahedra P, there is no volume formula for P of the form

$$a_1 I(P) + a_2 F(P) + a_3 E(P) + a_4 W(P) + a_5 = V(P).$$

where $I(P)$ is the number of interior lattice points, $F(P)$ is the number of lattice points on the interior of the faces, $E(P)$ the number of lattice points on the edges excluding the vertices and $W(P)$ the number of vertices.

Higher Dimensional Analog: Reeve tetrahedron

Proof.

Consider the tetrahedras with given vertices.

	Vertices	I	F	E	W	V
T_1	$(0,0,0),(1,0,0),(0,1,0),(1,1,1)$	0	0	0	4	1/6
T_2	$(0,0,0),(1,0,0),(0,1,0),(1,1,2)$	0	0	0	4	2/6
T_3	$(0,0,0),(1,0,0),(0,1,0),(1,1,2)$	0	0	0	4	3/6
...	...	...	...	...	...	...
T_r	$(0,0,0),(1,0,0),(0,1,0),(1,1,2)$	0	0	0	4	$r/6$

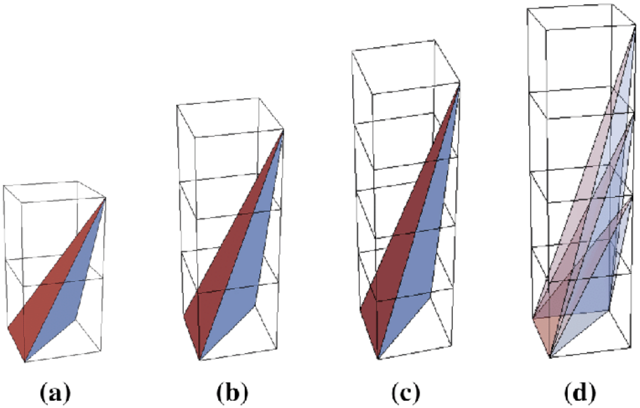
Then it results in the system of equations, for all r which is an integer.

$$4a_4 = r/6.$$

Therefore the system of equations are inconsistent, so there is no solution for a_1, a_2, a_3, a_4, a_5.

Note that these tetrahedras are called Reeve tetrahedras. □

Visualization: Reeve Tetrahedron



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Thanks

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